An Illustrated Introduction to the Ricci Flow

Gabriel Khan

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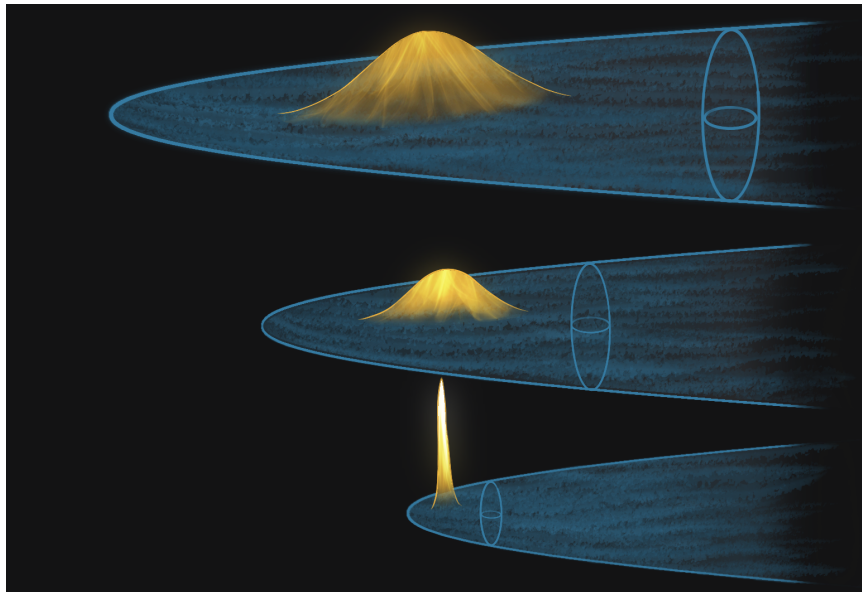

Figure 1: A Ricci flow with a conjugate heat flow[1]

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[1]This image depicts a Bryant soliton with a backwards heat flow. It is adapted from Figure 6 of [Bam21].

Contents

Introduction

The Ricci flow is one of the most important topics in differential geometry, and a central focus of modern geometric analysis. In this paper, we give an illustrated introduction to the Ricci flow for a general audience. In particular, we do not assume any background in differential geometry or differential topology, only that the reader has taken multivariate calculus. For those who are familiar with geometric analysis, we have provided additional details in the footnotes. However, these notes are not necessary to follow the main discussion, so can be ignored on an initial reading.

A short history of the Poincaré conjecture

Figure 2: A donut is topologically the same as a coffee mug

The Ricci flow is most famous for its role in the proof of the Poincaré conjecture, so we start our discussion with a historical account of the Poincaré conjecture and some related problems in topology.

In 1895, Henri Poincaré published the seminal paper *Analysis Situs* [Poi95], which laid the foundations for what is known today as *topology*.[2] Roughly speaking, topology studies "rubber-sheet geometry," where spaces are allowed to bend and stretch without tearing. Warping a space will change distances, areas, etc. so the properties studied in topology are quite

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[2]Some topological ideas appeared in Leonard Euler's earlier work on the Bridges of Konigsburg, but Analysis Situs is a foundational work in this area.

different from those in traditional geometry. However, one central focus is to find features which do not change when a space is deformed continuously.

While studying this topic, Poincaré wanted to understand three-dimensional spaces with the simplest possible topology. After several initial guesses which turned out to be false, he conjectured that the three-dimensional sphere is the only compact simply-connected three-dimensional manifold, which became known as the *Poincaré conjecture*. To explain the meaning of this conjecture, it is helpful to first consider the corresponding result for two-dimensional surfaces, which is a special case of the *uniformization theorem*.
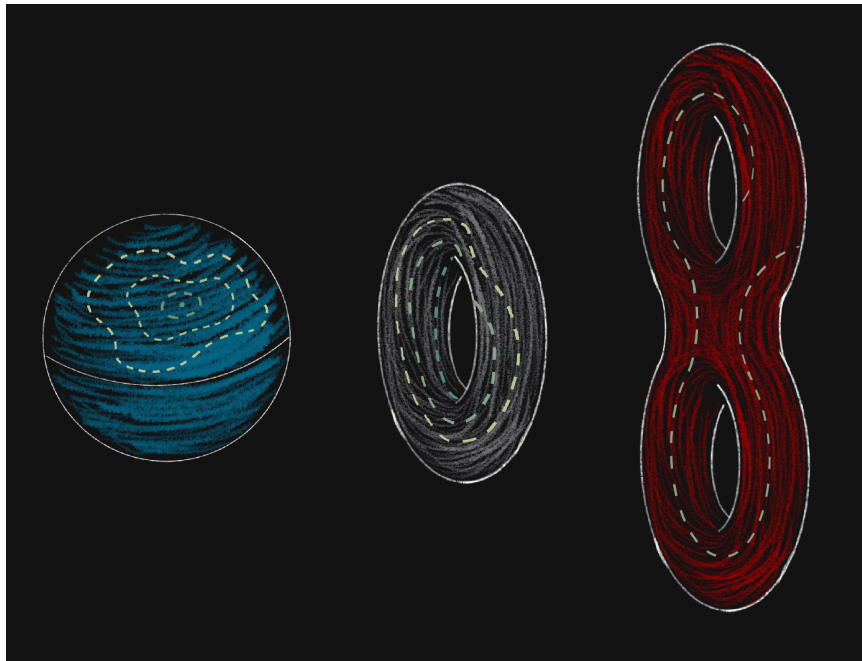
The uniformization theorem

Figure 3: Three surfaces and some loops on them[3]

The uniformization theorem is a celebrated result which classifies the possible geometries on a two-dimensional surface [Poi08]. It states any sur-

face can be deformed into one of three special types of geometries, depending on the number of holes that it has.[4] Surfaces without any holes can be deformed into a round sphere. A surface with one hole (i.e., a donut) can be made into a flat space[5] and any surface with more than one hole admits a hyperbolic geometry.

At first, it might be somewhat hard to visualize the latter two types of geometries. The reason for this is that when we think of surfaces, it is natural to visualize them as living in three-dimensional Euclidean space.[6] However, there is no way to put a flat donut or a hyperbolic surface in $\mathbb{R}^3$. To get around this roadblock, we must consider the *intrinsic* geometry of a surface, without assuming that it lies in some ambient Euclidean space. This can be a major conceptual challenge, but there is an intuitive way to visualize a flat donut using a classic arcade game.

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[4]The actual statement of the uniformization theorem is a bit stronger, and says that this deformation can be done in a conformal way. The theorem also gives a classification of non-orientable surfaces.

[5]Here, a surface is said to be *flat* if its curvature vanishes and hyperbolic if its sectional curvature is identically -1. Later on, we will give a definition for curvature.

[6]Here, by "living inside three-dimensional space," we mean smoothly immersed in $\mathbb{R}^3$. For readers who are familiar with the differential geometry of curves and surfaces, it is a good exercise to prove that compact surfaces with non-positive curvature cannot be immersed in $\mathbb{R}^3$. As a hint, suppose the surface contains the origin in its interior. What can we say about the curvature of the point furthest from the origin?
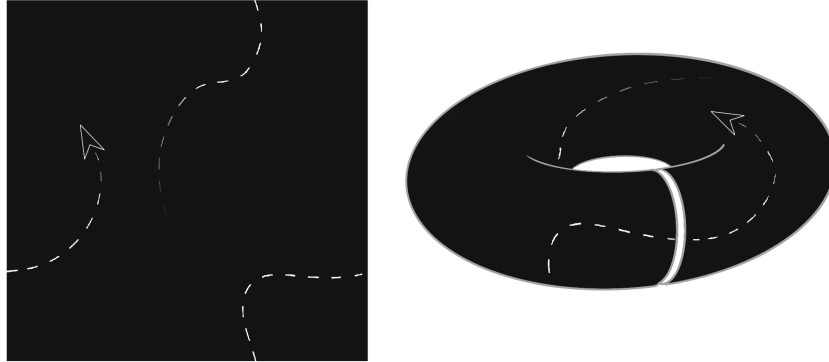
Figure 4: Asteroids on a donut

In the game Asteroids, the player pilots a ship through an asteroid field on the screen, but there is a catch. When the ship flies across the right edge of the screen, it ends up on the left side of screen and when it flies across the top, it ends up at the bottom. It is a worthwhile exercise to show that a space like this is topologically equivalent to a standard donut.

It turns out that there are flat donuts in four-dimensional Euclidean space. Furthermore, there is a very deep theorem by John Nash which states that any possible geometry on a surface can be realized if we consider the surface as living in 51-dimensional Euclidean space [Nas56].[7]

The Poincaré conjecture

The uniformization conjecture has a straightforward, but very important, corollary relating the topology of a surface and the behavior of loops on it. As shown in Figure 3, if we consider a surface with a positive number

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[7]In general, any n-dimensional Riemannian manifold can be embedded in $\mathbb{R}^N$ where $N = n(n+1)(3n+11)/2$. The dimension 51 comes from applying this formula to $n = 2$. Mikhail Gromov proved that any compact surface can actually be isometrically embedded in $\mathbb{R}^5$, so for our purposes 46 of these 51 dimensions are redundant [Gro86].
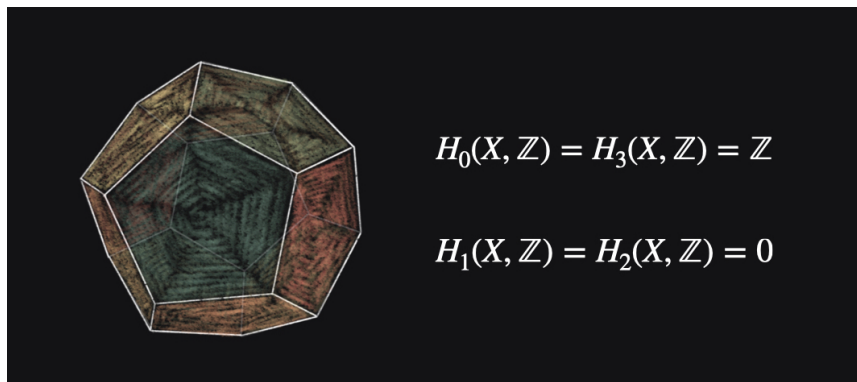
$$H_0(X, \mathbb{Z}) = H_3(X, \mathbb{Z}) = \mathbb{Z}$$

$$H_1(X, \mathbb{Z}) = H_2(X, \mathbb{Z}) = 0$$

Figure 5: A counter-example to an early version of the Poincaré conjecture[10]

of holes and draw a loop around the hole, it is not possible to shrink the loop to a point without cutting it or leaving the surface.

As a result, one consequence of the uniformization theorem is that if we are given a surface where every loop drawn in it can be shrunk down continuously to a point, then the space is topologically equivalent to a sphere. Spaces where every loop can be contracted to a point are said to be *simply-connected*, so another way to state this is that any simply-connected surface is topologically equivalent to a sphere. The *Poincaré conjecture* is the three-dimensional version of this statement. In other words, if we are given a closed three-dimensional space[8] which is bounded and simply-connected, then the space is topologically equivalent to a three-dimensional sphere.

This conjecture (and its higher dimensional analogues) motivated much of the early work in topology. The three-dimensional case in particular proved itself to be extremely subtle and intractable problem (see [Sta16] for some idea of why this is the case). It attracted the attention of many leading mathematicians who developed various tools in their attempts to solve it. As a result of all these efforts, by the 1980s the Poincaré conjecture was settled

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[8]Throughout the rest of the paper, we will use the word "space" to mean "compact smooth manifold" so as to not introduce unnecessary terminology.

[10]More precisely, if we take a solid dodecahedron and identify the opposite sides with a minimal twist, what results is a *homology sphere*, which is a manifold whose homology groups are the same as the sphere but which is topologically distinct from a sphere.

in all dimensions except for three.[11] However, Poincaré's original conjecture appeared to be outside the reach of the standard tools of low-dimensional topology, and it seemed that a new approach would be needed to solve it.

The Geometrization Conjecture

Figure 6: Thurston's Eight Geometries[12]

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[11]There is an important subtlety here. There is a both a topological and a smooth version of the Poincaré conjecture, which considers whether spaces are continuously equivalent to a sphere or smoothly equivalent to the round sphere. The former is what is traditionally known as the Poincaré conjecture. Stephen Smale proved the topological Poincaré conjecture in dimensions greater than four [Sma62] in 1962 and Michael Freedman proved the four-dimensional case [Fre82] in 1982. On the other hand, John Milnor showed that there are exotic smooth structures on the seven sphere, and thus the smooth Poincaré conjecture is false in dimension seven [Mil59], which started a long line of work study the possible smooth structures on spheres. All three were awarded Fields medals for their work. At this point, there is one major question remaining, which is whether there exist exotic smooth structures on the four-dimensional sphere.

[12]In this illustration, the top row illustrates the five geometries which are either spaces of constant curvature or products thereof. The illustrations of the bottom row deserve further explanation.

1. The depiction of Sol illustrates the metric

$$g = e^z dx^2 + e^{-z} dy^2 + dz^2,$$

7

In the mid-1970s, William Thurston began a line of work to understand the geometry of three-dimensional spaces, with an emphasis on those with hyperbolic geometry. His approach to studying geometry emphasized visualizing space and finding geometric structure, and yielded deep and unexpected insights.

Building from his work on hyperbolic spaces, he found eight standard geometries for closed three-dimensional spaces. With these geometries, he proposed a classification for the possible geometries of three-dimensional spaces similar to how the uniformization theorem classifies the geometry of two-dimensional surfaces. Unlike the case in two dimensions, it is possible to construct three-dimensional spaces which do not admit any *one* geometry. Instead, Thurston conjecture that given a closed three-dimensional space, it is possible to cut the space into pieces along two-dimensional spheres and donuts so that each of the resulting pieces admits one of these eight geometries. Thurston was able to prove this conjecture for a fairly broad class of spaces and was awarded the Fields Medal in 1982 for this work. His conjecture, known as Thurston's Geometrization conjecture, implied the Poincaré conjecture and gave a new avenue to attack Poincaré's question. Although the Geometrization conjecture was only a conjecture, it provided the first strong piece of evidence for why mathematicians should believe the Poincaré conjecture. Before this point, the best piece of evidence was that no one had managed to find a counter-example.

which expands exponentially in the x-direction and contracts exponentially in the y-direction as the z-coordinate increases.

2. The fundamental example of a space which has $\widetilde{SL}(2, \mathbb{R})$-geometry is the complement of the trefoil knot in the three-sphere.

3. Nil has the geometry of a torus bundle over a circle where the monodromy is given by a Dehn twist [CMST20]. Here, the vertical lines depict the universal cover of a circle so that the two tori appear at different heights.
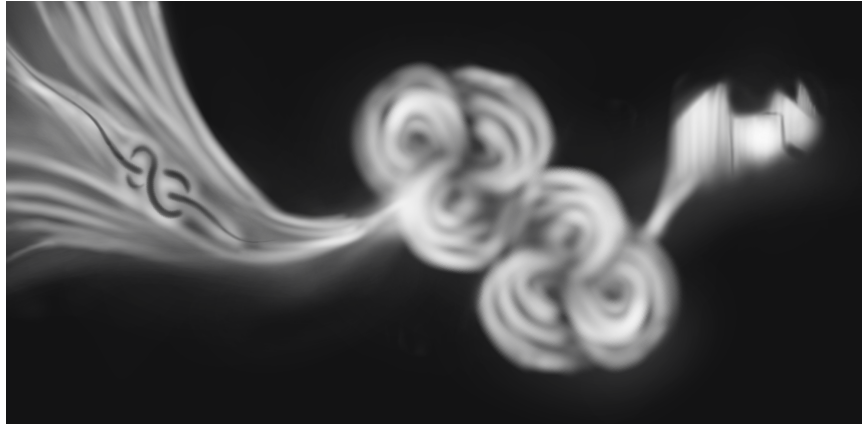
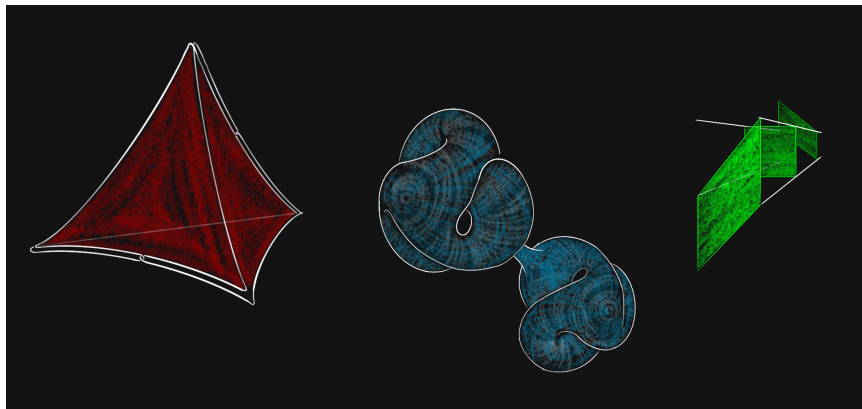Figure 7: A space formed from three different geometries...

Figure 8: And its geometric decomposition[13]

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[13]This figure incorporates some artistic license as the complement of a figure-eight knot is non-compact, so the space as depicted does not satisfy the hypotheses of the Geometrization conjecture. However, it seemed necessary to include at least one depiction of a hyperbolic knot complement while discussing Thurston's work. The image of the knot is adapted from the lower figure on page 5 of [Thu79]. The central component of this manifold is a $\mathbb{S}^2 \times \mathbb{R}$ structure on the connected sum of two copies of $\mathbb{RP}^3$. Since there is not a good way to draw three-dimensional projective space, what is shown is a connected sum of two Boy's surfaces, which are immersions of $\mathbb{RP}^2$ in $\mathbb{R}^3$. The right-most piece has Nil geometry, which is depicted by a portion of the Cayley graph of the Heisenberg group.

The Ricci flow

Around this time, Richard Hamilton proposed an ambitious program to attack both of the Geometrization and the Poincaré conjectures [Ham82]. He had been studying a paper by James Eells and Joseph Sampson [ES64] on harmonic maps. This paper was notable because it used a heat flow to find the harmonic maps in question, and Hamilton thought it might be possible to use a similar approach here.[14] He defined an evolution equation, known as the Ricci flow, which would deform the shape of a space and hopefully allow its curvature to dissipate throughout the space. If the space was simply connected, he hoped the geometry would evolve to that of a round sphere, which would establish the result.

Hamilton and others were able to make partial progress towards this goal. In particular, Hamilton showed that starting with a simply connected three-dimensional space whose Ricci curvature is positive, it would indeed evolve to a round sphere. This was a weaker version of the Poincaré conjecture, but a major proof of concept for the power of the Ricci flow. Furthermore, he was nearly able to find a new proof of the uniformization theorem using Ricci flow, and this proof was completed by Bennett Chow [Cho91] and Xiuxiong Chen, Peng Lu, and Gang Tian [CLT06]. These results suggested that the Ricci flow was a promising approach to the Poincaré conjecture, but there was a problem. The flow would encounter *singularities*, which are times where the space would either collapse to a point[15] or violently rip itself apart. Hamilton understood the cases when the space would shrink to a point, but was was not able to control the geometry in the latter case, and this presented a fundamental obstacle to finishing the proof.

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[14]Strictly speaking, the Ricci flow is not a (non-linear) heat flow. In particular, it is diffeomorphism-invariant, which induces zero terms in its symbol. As a result of this, establishing the existence for the flow was a major accomplishment. Hamilton did so via a technical argument [Ham82] but the next year Dennis DeTurck found a much simpler argument by conjugating the Ricci flow to obtain a strictly parabolic flow [DeT83].

[15]The Ricci flow can also collapse in other ways. For instance, if we consider the Cartesian product of a round sphere with a unit circle (with the product metric), Ricci flow will collapse the space $\mathbb{S}^2 \times \mathbb{S}^1$ to the unit circle.
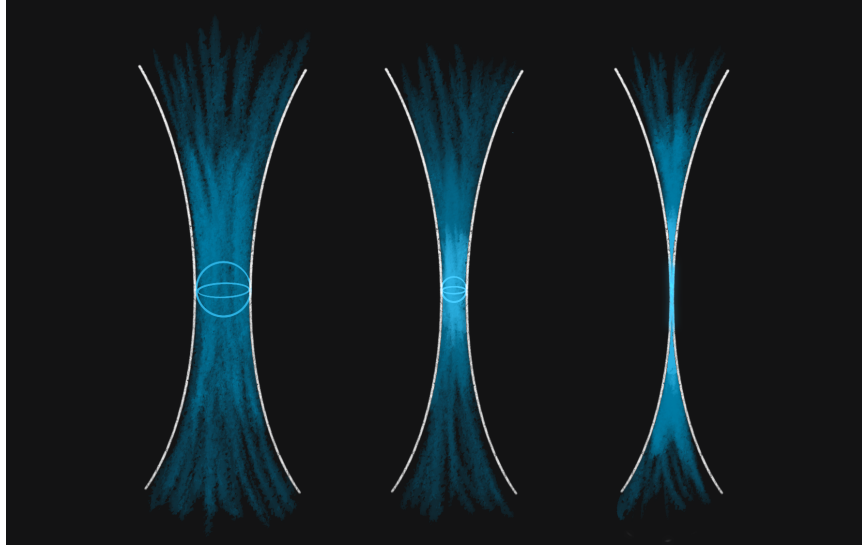
Figure 9: The Ricci flow becoming singular

Perelman's breakthrough

In 2000, the Clay Millenium institute listed seven major open problems in mathematics and provided a $1,000,000 prize for a solution to any one of them. The Poincaré conjecture was chosen as one of the problems and despite the best efforts of many mathematicians to solve it, no one expected to see a solution in the near future.

However, just two years later, Grigori Perelman posted a preprint to the arXiv [Per02] with several major breakthroughs in the Ricci flow. In particular, he showed that by coupling the Ricci flow with a heat equation running in reverse, there was a non-decreasing quantity (which corresponds to the *Fisher information* of the heat distribution[16]) which could be used to control the geometry of its solutions. He also defined a distance on *Ricci flow space-time* and a volume associated with this distance. Using these quantities, Perelman classified the potential singularities which appear in three-

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[16]For background on the Fisher information, we refer the reader to Chapter 20 of Cedric Villani's textbook Optimal Transport, old and new [Vil09].

dimensions and sketched a proof for both the Poincaré and Geometrization conjectures. This paper took the geometry community by storm, and the experts studied it to understand what Perelman had found.
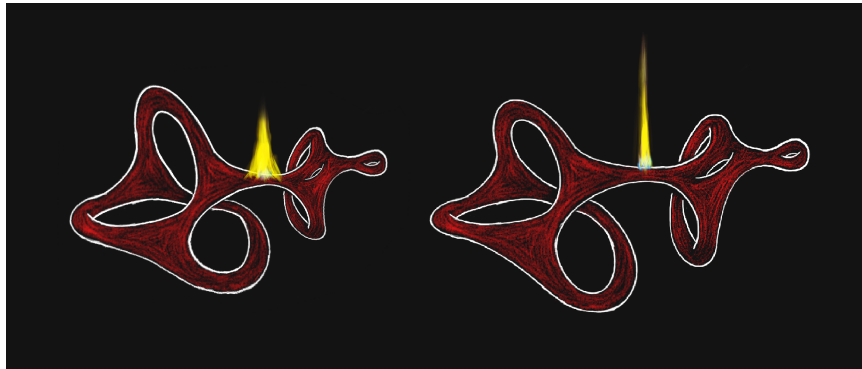

Figure 10: An evolving surface with a backwards heat flow

Over the eight months, two more preprints [Per03b, Per03a] followed the first and provided more details. In particular, Perelman combined the ideas from the first paper with a geometric process known as surgery to excise regions of space when they started to tear apart and replace them with pieces with better behavior.[17] By combining the Ricci flow with surgery, Perelman was able to show that any simply-connected three-dimensional space would converge to a round sphere (or possibly several round spheres which could be reconnected), and thus is topologically equivalent to the standard sphere. For more general three-dimensional spaces, he proved that after a large amount of time, Ricci flow with surgery would decompose the space into several pieces whose geometric structure was well understood.[18] This established both the Poincaré and Geometrization Conjectures.

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[17]Ricci flow with surgery was invented by Hamilton in 1993 [Ham93, Ham97], but he was unable to use it to handle the singularities in the three dimensional case.

[18]Ricci flow with surgery does not necessarily converge to one of Thurston's geometries on each connected component. Several of the geometries collapse under the flow so are not even fixed points. Instead, it is now known that if we perform the surgeries in an effective way, there are only finitely many of them [Bam18], and after the last one the space decomposes into finitely many pieces, all of which were previously known to satisfy the Geometrization conjecture [Cal20].
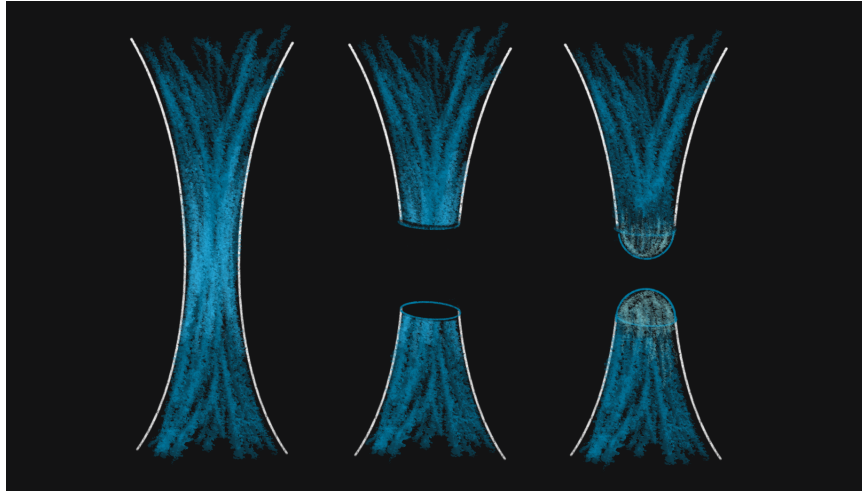
Figure 11: Using surgery to tame singularities

Perelman had spent nearly a decade working in relative isolation, and his results were the crowning achievement in a long line of work stretching back hundreds of years. The proof sparked an enormous amount of interest in geometric analysis and the Ricci flow in particular. However, it took several years (and a authorship scandal) for the mathematical community to accept the work as correct. In 2006, Perelman was awarded a Fields medal, but declined the award and later turned down the Millenium Prize, as well. He has since withdrawn from mathematics.

What is the Ricci flow?

The goal of this section is to provide a working definition for the Ricci flow with some intuition for its behavior. Heuristically, the Ricci flow is a "heat flow of curvature" and we will try to explain what this means. To do so, we have divided this section into four subsections.

The Heat Equation

The heat equation models how the temperature of a region changes in time. More precisely, we consider a function $u(\mathbf{x}, t)$ which returns the temperature of a point[19] $\mathbf{x} \in \mathbb{R}^n$ at a time t. Ignoring what happens at the boundary of the region,[20] we say that the function $u(\mathbf{x}, t)$ solves the heat equation if it satisfies

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} = \Delta u(\mathbf{x}, t). \tag{1}$$

This equation was introduced by Joseph Fourier in 1822, and played a foundational role in the development of Fourier analysis. We will not provide a

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[19]Here we use boldface $\mathbf{x}$ to distinguish from the x-coordinate.

[20]In the context of the Ricci flow, we will be dealing with compact manifolds without boundary so this is not an issue.
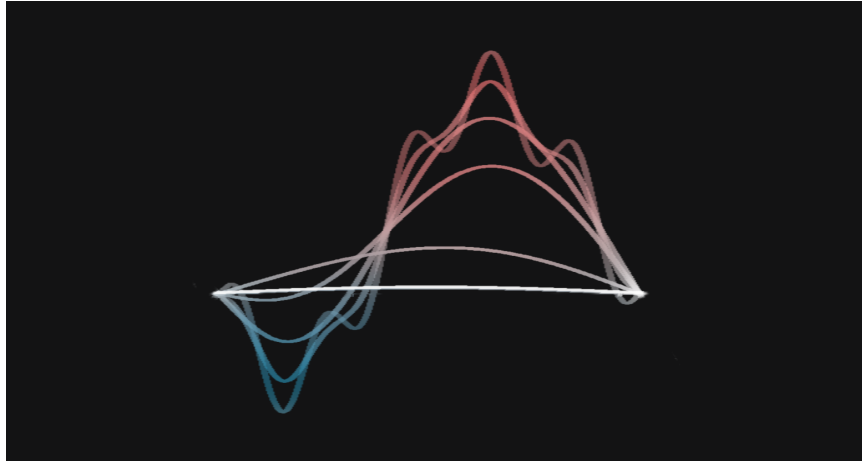
Figure 12: A heat distribution flowing to equilibrium[21]

physical derivation for why heat distributions tend to obey this equation, but instead try to understand the behavior of its solutions.

The Laplacian

Before trying to understand the behavior of solutions to Equation 1, let us first try to understand each of its terms. The left hand side of Equation 1 is the partial derivative of u with respect to t, which describes how the temperature at a point $\mathbf{x}$ evolves as time goes by. Therefore, we must understand the term Δu, which is the Laplacian of u. In vector calculus, this is often defined in the following way:

$$\Delta u = \text{div}(\text{grad } u).$$

However, there are several other perspectives on the Laplacian that will tie in more naturally to Ricci flow. One useful interpretation of the Laplacian

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[21]In this diagram, we have imposed Dirichlet boundary conditions on the solution. The time slices were chosen to show the qualitative behavior of the solution, and were not taken uniformly.

is to consider the (three-dimensional) Hessian matrix

$$
\mathrm{Hess}(u) =
\begin{bmatrix}
\frac{\partial^2}{\partial x^2} u & \frac{\partial}{\partial x}\frac{\partial}{\partial y} u & \frac{\partial}{\partial x}\frac{\partial}{\partial z} u \\
\frac{\partial}{\partial y}\frac{\partial}{\partial x} u & \frac{\partial^2}{\partial y^2} u & \frac{\partial}{\partial y}\frac{\partial}{\partial z} u \\
\frac{\partial}{\partial z}\frac{\partial}{\partial x} u & \frac{\partial}{\partial z}\frac{\partial}{\partial y} u & \frac{\partial^2}{\partial z^2} u
\end{bmatrix}
$$

and note that the Laplacian is the trace of this matrix:

$$
\Delta u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u + \frac{\partial^2}{\partial z^2} u.
$$

Later on, we will see that the Ricci curvature is the trace of the Riemann curvature tensor, so this identity makes the connection between the Ricci curvature and the Laplacian fairly natural.
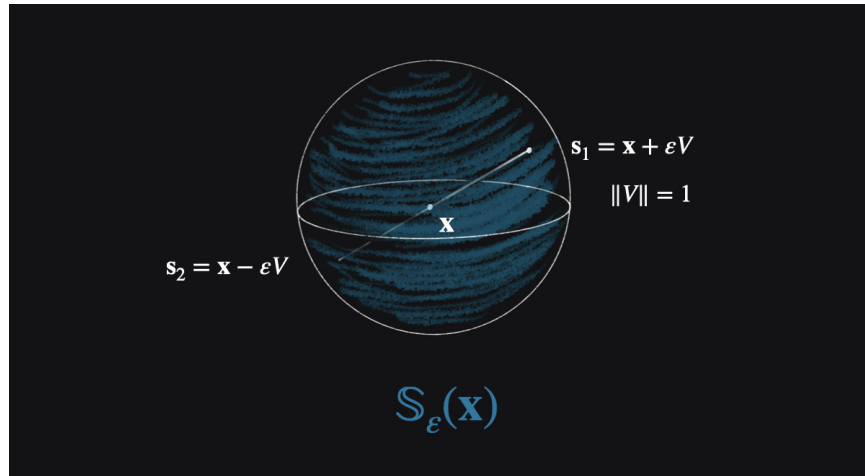

Figure 13: The Laplacian as an integral

In order to define the Ricci flow using the minimal amount of Riemannian geometry, there is a third perspective on the Laplacian that is even more useful. We consider a point $\mathbf{x}$ and a small sphere of radius ε around $\mathbf{x}$, which we denote $\mathbb{S}_\varepsilon(\mathbf{x})$. One can show that the Laplacian is proportional to the difference between $f(\mathbf{x})$ and the average value of f on $\mathbb{S}_\varepsilon(\mathbf{x})$:

16

$$\Delta f(\mathbf{x}) = \lim_{\varepsilon \to 0+} \frac{2n}{\varepsilon^2} \frac{1}{\omega\left(\mathbb{S}_\varepsilon\right)} \int_{\mathbb{S}_\varepsilon(\mathbf{x})} \left(f(\mathbf{s}) - f(\mathbf{x})\right) \mathrm{d}\omega(\mathbf{s}). \tag{2}$$

Here, $\omega\left(\mathbb{S}_\varepsilon\right)$ is the surface area of a sphere of radius ε in Euclidean n-space, which is given explicitly by

$$\omega(\mathbb{S}_\varepsilon) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \varepsilon^{n-1}.$$

From this we can see that Δu is the average of second derivatives, which is a perspective that will help us understand Ricci curvature. It is worth convincing yourself that the right-hand side of (2) really does have something to do with second derivatives.[22]

Convergence to equilibrium

One hallmark of the heat equation is that its solutions tend to converge to an equilibrium. That is to say, given a function $u(\mathbf{x}, t)$ which solves the heat equation, we expect that

$$\lim_{t \to \infty} u(\mathbf{x}, t) = C$$

where C is a constant which is independent of the point. Intuitively, if we have a warm cup of coffee outside in a cold day, this states that the heat from the coffee will dissipate into the air and the temperature will converge to that of the outside environment.

There are various ways to make this idea into a precise mathematical statement, and we will mention two that are relevant for understanding the Ricci flow.

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[22]Figure 13 is meant to serve as a hint for why this is the case. As an additional hint, given a unit vector V, what is

$$\lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon V) - f(\mathbf{x}) + f(\mathbf{x} - \varepsilon V) - f(\mathbf{x})}{\varepsilon^2}?$$

The maximum principle

One of the most fundamental tools that one has for studying heat equations is the *maximum principle.* To give a simple illustration of this principle, we will show that the hot spots of $u(\mathbf{x}, t)$ tend to cool down while the cold spots tend to warm up. To do so, we suppose the temperature u is maximized at the point $\mathbf{x}_0$ (in the interior of the domain) at some time t. Then, since $u(\mathbf{x}_0, t)$ is maximized, the second derivative test implies that the Hessian of u at $\mathbf{x}_0$ must be non-positive definite (i.e., have all non-positive eigenvalues). This implies that $\Delta u(\mathbf{x}_0, t)$, which is the trace of the Hessian must also be non-positive. From the heat equation, this forces that

$$\frac{\partial u(\mathbf{x}_0, t)}{\partial t} \leq 0,$$

which is to say that the temperature at $\mathbf{x}_0$ is decreasing. If we consider the coldest points, the same argument shows that they warm up under the heat flow.

This type of argument, where one considers a point which maximizes some function and applies the second derivative test (or a more sophisticated version of it) is endlessly useful, and plays an essential role in the analysis of heat-type equations. For example, one can use this argument to prove the Li-Yau estimate [LY86], which states that positive solutions to the heat equation on a space with non-negative Ricci curvature satisfies the inequality

$$\Delta \ln u(\mathbf{x}, t) \geq -\frac{n}{t}.$$

This particular estimate plays a fundamental role in geometric analysis, including in the study of the Ricci flow.

Entropy and energy

Another fundamental tool for heat-type equations is to find quantities like energy or entropy which either increase or decrease along the flow. By doing so, one can often control the behavior of the heat equation.
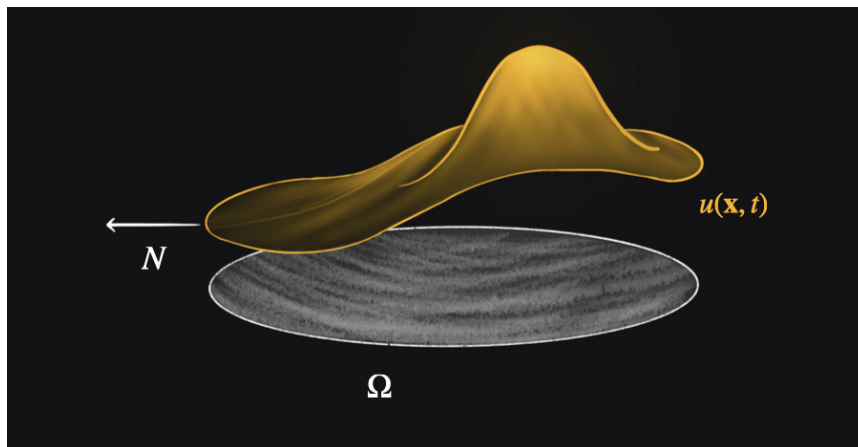
Figure 14: A perfectly insulated domain

To show this idea in practice, we consider the heat equation on a smooth bounded domain Ω and assume that the domain is perfectly insulated from its surroundings. From a mathematical perspective, this assumption is equivalent to requiring that the temperature at the boundary of the domain satisfies

$$\nabla_N u(\mathbf{x}, t) = 0$$

where N is the outer normal vector, which is known as *Neumann boundary conditions*.[23]

We then consider the quantity

$$E(t) = \int_\Omega u(\mathbf{x}, t)^2 \, d\mathbf{x}, \tag{3}$$

which is often called the *energy*. However, this doesn't have a direct physical interpretation so the name should not be taken too literally. Intuitively, the energy measures how concentrated the heat distribution is. In other words, if $u(\mathbf{x}, t)$ is very large in some region $\mathbf{x}$, $E(t)$ will also be large. On the other hand, if the distribution is very diffuse, $E(t)$ will be much smaller. We can compute how $E(t)$ evolves in time as follows.

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[23] For those who have taken a class on PDEs, it is a good exercise to show that the total heat is conserved under Neumann boundary conditions.

$$\frac{dE}{dt} = \int_\Omega \frac{\partial}{\partial t} u(\mathbf{x}, t)^2 \, d\mathbf{x} \tag{4}$$

$$= \int_\Omega 2 \left(\Delta u(\mathbf{x}, t) \right) \cdot u(\mathbf{x}, t) \, d\mathbf{x}. \tag{5}$$

Using integration by parts and the generalized Stokes' theorem, it is possible to simplify this expression.

$$\frac{dE}{dt} = -\int_\Omega 2\|\nabla u(\mathbf{x}, t)\|^2 \, d\mathbf{x} + \int_{\partial\Omega} 2u(\mathbf{x}, t) \left\langle \nabla u(\mathbf{x}, t), N(\mathbf{x}) \right\rangle \, d\mathbf{x}. \tag{6}$$

However, the boundary conditions implies that the second term vanishes.

$$\frac{dE}{dt} = -\int_\Omega 2\|\nabla u(\mathbf{x}, t)\|^2 \, d\mathbf{x} < 0.$$

As such, the energy decreases along the flow, which shows that the heat distribution tends to spread out through space. With a bit more effort, it is also possible to show that this quantity is convex.

Another quantity that we can consider is

$$S = -\int_\Omega u(\mathbf{x}, t) \ln \left(u(\mathbf{x}, t) \right) \, d\mathbf{x},$$

which is better known as the *entropy*.[24] The entropy is a measure of how disordered a configuration is. More precisely, it is the amount of information (measured in *nats*) that the heat distribution contains relative to the equilibrium state. Entropy is a notoriously tricky concept to conceptualize, so for our purposes we can simply define it using the integral above. We can use the same idea from before to calculate the evolution of S. Doing so, we find the following:

$$\frac{dS}{dt} = \int_\Omega \frac{\|\nabla u\|^2}{u} \, d\mathbf{x} \tag{7}$$

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[24]Here, we need to assume that the temperature is everywhere positive for this quantity to be well-defined.

The quantity on the right hand side of this equation is known as the Fisher information, and is positive. As such, Equation 7 is a mathematical version of the second law of thermodynamics, which states that a closed system tends to go from an orderly configuration to a disordered state. These types of quantities play a central role in the Ricci flow and one of Perelman's most important breakthroughs was to find a version of "entropy" for the Ricci flow.[25]

Curvature

In mathematics, curvature is a way to measure how much a shape of space bends. For instance, the curvature of a curve is defined as how quickly it turns as we travel along it with unit speed. It turns out that this curvature is precisely one divided by the radius of the circle which best approximates the curve at that point.[26]
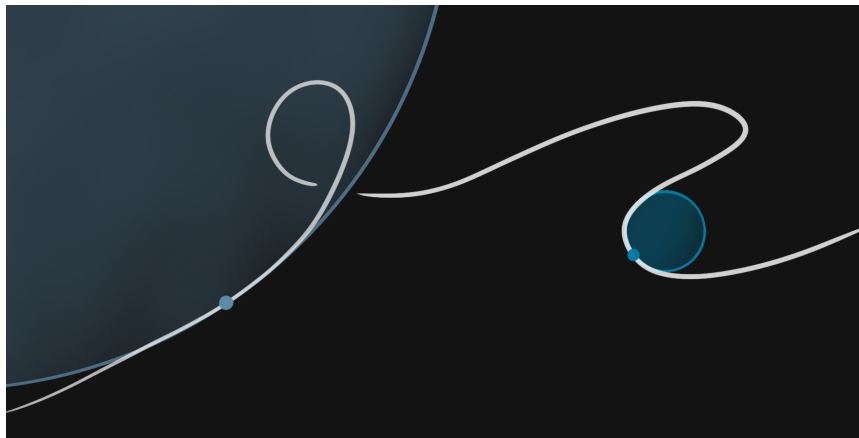

Figure 15: A curve with points of large and small curvatures

There are many different types of curvatures, and these notions appear

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[25]In fact, his first paper in the trilogy is titled "The entropy formula for the Ricci flow and its geometric applications" [Per02]. However, the precise quantity is more similar to a Fisher information rather than an entropy.

[26]This notion of curvature corresponds to the geodesic curvature of the curve.

throughout geometry and physics. For example, the shape of a soap film is determined by one particular type of curvature known as the *mean curvature.*

In the theory of general relativity, *Einstein's field equations* state that the curvature of space-time is determined by how much matter and energy is present. On earth, this curvature is minimal, which is why we don't notice it in our day-to-day lives.
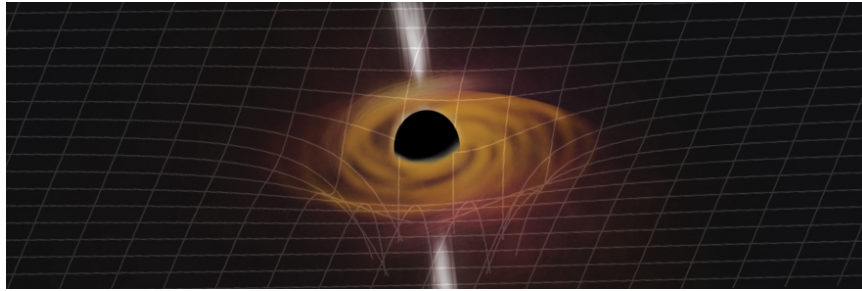

Figure 16: A black hole curving space-time

However, when huge amounts of matter accumulate in a small region (such as what occurs in the aftermath of a supernova), the curvature can become so large that it tears the very fabric of space-time and creates a *black hole.* Black holes are examples of singularities for the Einstein field equations. However, there are some important technical differences between the Einstein field equations and Ricci flow, so singularities of Ricci flow are actually quite different from black holes.

Unfortunately, rigorously defining the curvature which is relevant for the Ricci flow requires some background knowledge in Riemannian geometry and a more in-depth discussion of extrinsic versus intrinsic geometry, which are both outside the scope of this introduction. As such, in this section we will provide an intuitive notion of curvature, without trying to be overly precise.

Sectional curvature

It is not immediately obvious what curvature is, especially from an intrinsic perspective where the space of interest does not live in some ambient Euclidean space. However, one useful intuition is that Euclidean space is

flat, without any bumps or valleys. Using this idea, one way to formalize the notion of curvature is to compare the geometry of a given space with that of Euclidean space. To do so, the simplest approach is to consider triangles.
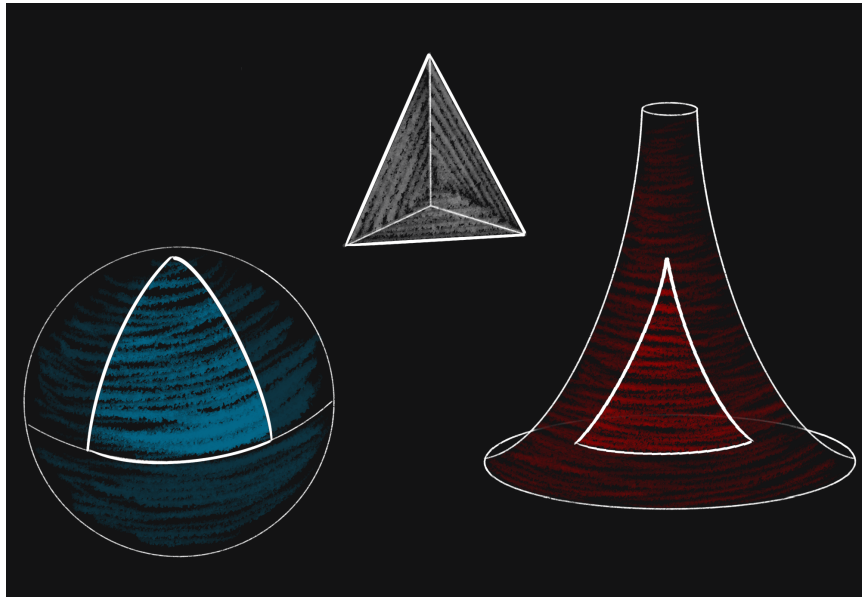

Figure 17: Triangles in spherical, flat and hyperbolic space

As can be seen in Figure 17, triangles on a sphere[27] appear "fatter" than triangles in flat space. There are a few ways to make this precise. For instance, we can consider the sum of the angles of a spherical triangle. It turns out the sum is greater than π, and actually depends on the area of the triangle.[28] Apart from the angles, we can also see from the picture that the sides of a spherical triangle seem to bow away from the other sides. This bowing occurs in any space of positive curvature, and is one of the characteristic features of positive sectional curvature. To understand this,

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[27]More precisely, we are considering geodesic triangles, where each side is length-minimizing.

[28]For readers who have studied spherical trigonometry, this will be a familiar fact. On a two dimensional sphere, it is actually a consequence of the Gauss-Bonnet theorem and for those who are familiar with the differential geometry of curves and surfaces, it is a good exercise to compute the exact formula for the sum of the angles.

it helps to picture the sides as turning towards each other, which is why they appear to bow outward. Here, the meaning of the word "turning" is somewhat imprecise, but hopefully the picture makes it clear what we mean.

Before we use this idea to discuss curvature, it is worthwhile to note that round spheres are quite special, in that every point looks like every other point.[29] However, the earth is not exactly a sphere; the geometry of Mount Everest looks very different from that of the Great Plains. This will be true for most of the spaces we are interested in, so we want some way to define curvature on spaces which are not so symmetric. To do so, instead of using a giant triangle like on our sphere, we use very small ones.

To do this, we consider a point in our space and two tangent vectors X and Y. Then we consider a triangle which has one vertex at p and which has two sides of length ε in the directions of X and Y. We will not be too precise about what it means to make a triangle where the sides "head in a direction," because it takes some work to define.[30] On the sphere, we saw that the triangle will turned in on itself as a result of the positive curvature. With this in mind, we consider the length of the third edge of the triangle, which we denote $L(\varepsilon)$.
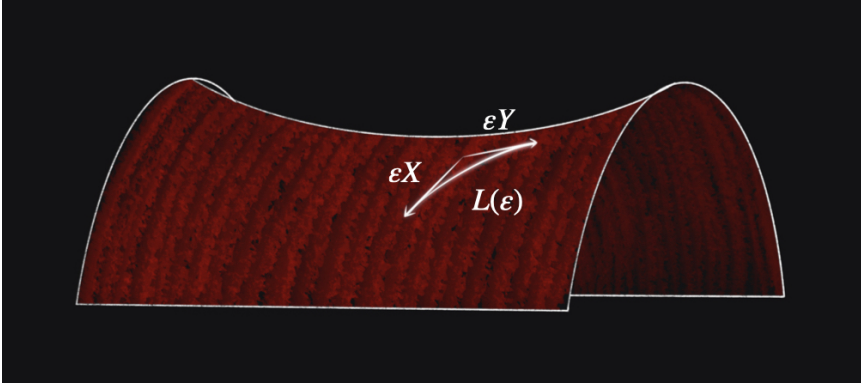

Figure 18: A heuristic diagram for $L(\varepsilon)$

The curvature will be positive whenever $L(\varepsilon)$ is smaller than it would

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[29]More precisely, the round sphere is a space with *constant curvature*.

[30]More precisely, we take a tangent vector X and consider the geodesic $\gamma(t) = \exp_p(Xt)$.

be for a corresponding triangle[31] in Euclidean space and negative if $L(\varepsilon)$ is greater than a triangle in Euclidean space. To make this idea more formal, we consider the Taylor polynomial of $L(\varepsilon)$ in terms of ε. Doing so, we find the following:

$$L(\varepsilon) = \varepsilon\|X - Y\| \left(1 - \frac{1}{12}K(X,Y)(1 + \langle X,Y \rangle)\varepsilon^2\right) + O\left(\varepsilon^4\right) \qquad (8)$$

In this expression, $K(X,Y)$ is defined to be the sectional curvature of the tangent plane spanned by X and Y. Note that in flat space, $K(X,Y) \equiv 0$ so this matches with our intuition. On the other hand, the unit sphere has constant positive sectional curvature (i.e., $K(X,Y) \equiv 1$). Since $K(X,Y)$ is positive, the third side of the triangle will be shorter than the corresponding side in flat space. Conversely, whenever the sectional curvature is negative, the third side of the triangle will be longer than that of the corresponding Euclidean triangle.

Ricci curvature

While the sectional curvature contains all of the curvature information about a space, it is a very complicated object. The Ricci curvature is a coarser invariant than the sectional curvature,[32] but conveys important geometric information that is essential in many applications. To define it, we consider a unit vector X and define the Ricci curvature[33] $\mathrm{Ric}(X,X)$ to

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[31]Here, a *corresponding triangle* means a triangle $\triangle PQR$ whose angle $\angle RPQ$ is the equal to the angle between X and Y and where the sides $\overline{PQ}$ and $\overline{PR}$ have length ε.

[32]In dimensions two and three, the Ricci curvature determines the sectional curvature completely, but this is not the case in higher dimensions.

[33]The standard definition of the Ricci curvature is as the contraction of the Riemann curvature tensor along second and last indices. In other words, given an orthonormal frame $\{e_i\}_{i=1}^{n}$ and two vector fields X and Z,

$$\mathrm{Ric}(X,Z) = \sum_{i=1}^{n} \langle R(X,e_i)Z, e_i \rangle$$

Here, $R(\cdot,\cdot)\cdot$ is the Riemannian curvature tensor, which is defined as

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]}Z.$$

$(n-1)$ times the average of all of the sectional curvatures of tangent planes containing X. In other words, the Ricci curvature satisfies the identity[34]

$$\text{Ric}(X,X) = \frac{1}{2}(n-1) \oint_{\|Y\|=1 \text{ and } X \perp Y} K(X,Y) \, \mathrm{d}\mathbb{S}^{n-2}(Y), \qquad (9)$$

where $\mathrm{d}\mathbb{S}^{n-2}$ is the unit measure on the $(n-2)$-dimensional sphere. Initially, it might seem a bit strange that the vector X appears twice as an argument for the Ricci curvature, but we will just treat this as a convention without going into detail about why this is the case.[35]

Equation (9) gives a concise definition for the Ricci curvature, but does not provide any intuition for what the Ricci curvature actually is. To do so, it is possible to draw a picture for the Ricci curvature which is similar to the one for sectional curvature but uses cones rather than triangles.[36]

We consider a point p, a unit vector X at p and take a short segment of length ε in the X direction.[37] Finally, we put a narrow circular cone with vertex p around the segment (here, the cone being narrow means that the cone angle θ satisfies $\tan(\theta) = \varepsilon$). We then consider the area of the base of the cone, which we denote $A(\varepsilon)$.

---

where ∇_X denotes covariant derivative in the X direction with respect to the Levi-Civita connection. We used Equation (9) to avoid having to define the notions of Riemannian curvature and covariant derivatives.

[34]Note that the pairs (X,Y) and $(X,-Y)$ span the same plane, which is why there is a factor of $\frac{1}{2}$ in Equation 9.

[35]At each point, the Ricci curvature is a symmetric bilinear form, which is why there are two copies of X. To obtain the bilinear form from the average of sectional curvatures, one uses the polarization identity

$$\text{Ric}(X,Y) = \frac{1}{2}\left(\text{Ric}(X+Y,X+Y) - \text{Ric}(X,X) - \text{Ric}(Y,Y)\right)$$

.

[36]The idea of Ricci curvature as the distortion of narrow geodesic cones is taken from Chapter 14 of Villani's textbook on optimal transport [Vil09].

[37]More precisely, we consider a geodesic γ of length ε with $\dot{\gamma}(0) = X$.
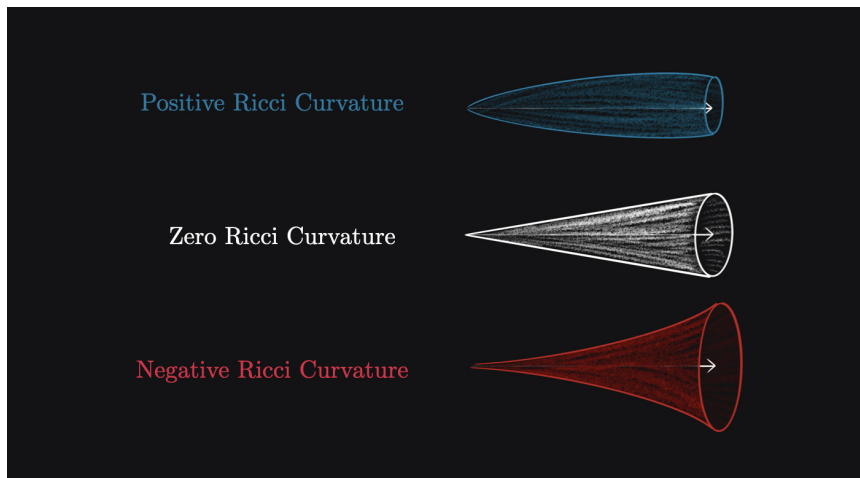
Figure 19: Cones with various Ricci curvatures

When the Ricci curvature is positive, the cone will close in on itself whereas in negatively curved space, the cone will open outward (as shown in Figure 19). In this spirit, when we compute the Taylor expansion of $A(\varepsilon)$, we find the following:

$$A(\varepsilon) = \varepsilon^{2(n-1)} \left(D_{n-1} - C_n \operatorname{Ric}(X,X)\varepsilon^2 \right) + O(\varepsilon^{2n+1}). \qquad (10)$$

Here D_{n-1} is the volume of a unit disk in Euclidean $(n-1)$-space and C_n is a complicated, but positive, constant which depends on the dimension. The factor of $\varepsilon^{2(n-1)}$ on the right-hand comes from the fact that the cone is both short and narrow, both of which contribute to the area of the base being small.

The Ricci curvature as a geometric Laplacian

Heuristically, the sectional curvature can be understood as a geometric second derivative. In particular, ignoring the ε at the front of Equation (8) (which comes from the fact that the associated triangle has very short sides), the sectional curvature appears in the second-order position. With this intuition, comparing Equations (2) and (9) suggests that the Ricci curvature is a sort of geometric Laplacian. Making this heuristic rigorous requires a

bit of Riemannian geometry, because the sectional curvature $K(X, Y)$ does not depend linearly on X and Y.

We won't go into too much detail about how to make this idea precise. The basic idea is that the sectional curvature can be used to construct the *Riemann curvature tensor*, which is comparable to a geometric Hessian.[38] The Ricci curvature is the trace of the Riemann curvature tensor, which gives further credence to its interpretation as a geometric Laplacian.[39]

There is one other important analogy between the usual Laplacian and the Ricci curvature, which is that they both measure how volumes change. More precisely, given a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, we can consider the gradient flow

$$\frac{d}{dt}\gamma(t) = -\nabla f(\gamma),$$

which takes a point and moves in the direction which decreases f the quickest. Doing so, the quantity $\Delta f(\mathbf{x})$ measures how much the volume of a very small cube around $\mathbf{x}$ will change under the flow. In other words, if we consider ∇f as the current of some fluid, Δf measures the compression of the flow. In a similar way, the Ricci curvature determines how volumes of small cubes change as we move from one point to another on a curved space.[40] In

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[38]It is possible to make the idea of the curvature tensor as a geometric second derivative reasonably precise. In particular, in any set of geodesic normal coordinates, the Riemannian metric satisfies

$$g_{ij} = \delta_{ij} - \sum_{k,l=1}^{n} \frac{1}{3} R_{ikjl} x^k x^l + O\left(|x|^3\right)$$

where R_{ijkl} denotes the components of the $(4, 0)$ Riemann curvature tensor. The key intuition from this formula is that it strongly resembles a second-order Taylor polynomial for a multivariate function, and the terms $-\frac{1}{3}R_{ijkl}$ correspond to the second-order terms in the expansion.

[39]There are other ways to make this precise. For instance, in harmonic coordinates,

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms...}$$

[40]The Weyl tensor is another curvature tensor which is orthogonal to the Ricci curvature and measures the "tidal forces." In other words, the Weyl tensor determines how the shape of small objects deform when they move along short geodesics whereas the Ricci curvature measures the compression of the gradient flow.
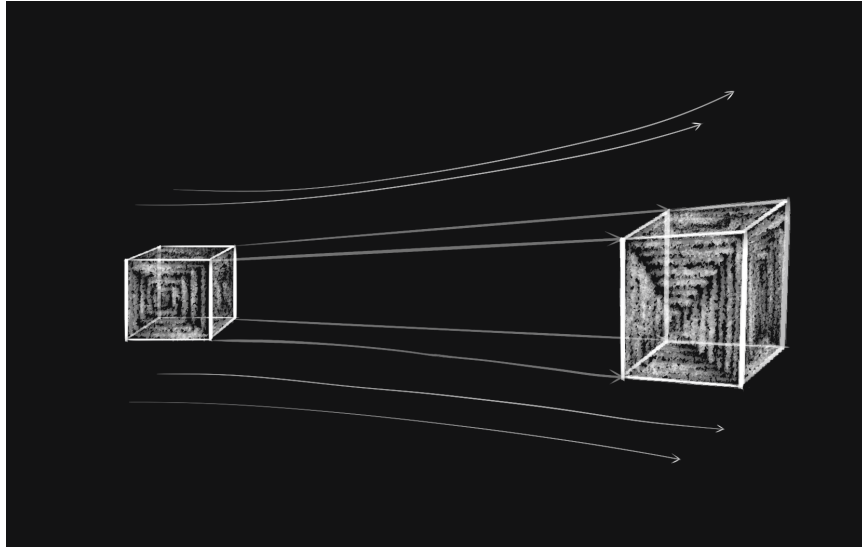
Figure 20: The compression of a fluid flow

other words, it measures how volumes are compressed due to the curvature of the space.

The Ricci Flow

Now that we have discussed the heat equation as well as the notion of Ricci curvature, we can finally talk about the Ricci flow, which can be understood as a "geometric heat equation." To get started, we will provide an informal geometric explanation.

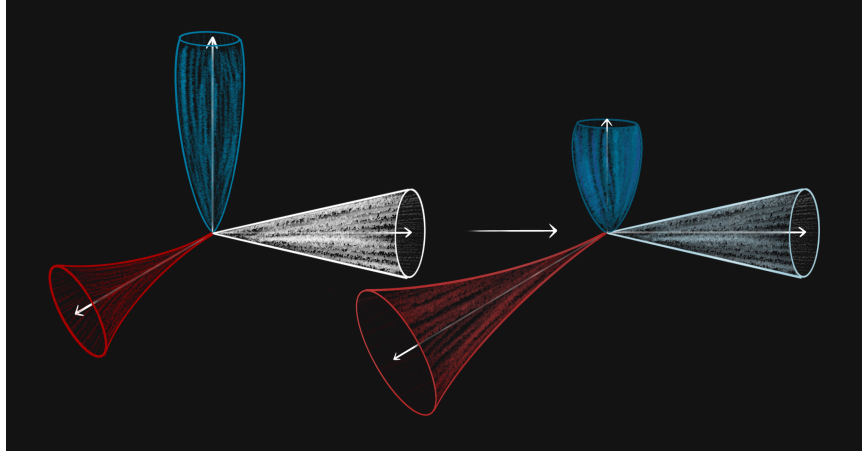**Definition.** *The Ricci flow changes the shape of a space proportional to -2 times the Ricci curvature.*

Figure 21: Ricci flow deforming a basis which is initially orthonormal[41]

In other words, directions which have negative Ricci curvature get longer whereas directions with positive Ricci curvature get shorter (as depicted in Figure 21).

In order to define the Ricci flow precisely, we must formalize what the "shape" of a space is. For surfaces in Euclidean space, it is clear enough what this refers to, but it is much more challenging to define the shape *intrinsically* when it is not lying in some higher dimensional space. To do so, we can use the notion of a Riemannian metric, which is a generalization of the dot product in Euclidean space. In other words, it provides an inner product of two tangent vectors (at the same point) in our space.[42] There is a fundamental theorem in differential geometry that states that this structure fully determines the geometry of the space, and that it is possible to compute the curvature of the space (as well as all the other metric invariants) from this inner product alone. However, the formulas involved are very complicated, so we will just consider the Riemannian metric g as a geometric object that

---

[41]Under Ricci flow, the curvature evolves in a complicated way, which is why the colors of the cones are also changing.

[42]More precisely, given a smooth manifold, a Riemannian metric g is smoothly-varying collection of positive-definite inner products on the tangent space of each point. It is worth noting that a Riemannian metric is not the same as a distance function, which is also known as a *metric*.

encodes the "shape" of our space. Using the metric, the equation for the Ricci flow becomes

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}(g). \tag{11}$$

It is worth spending some time to make sure this is really a sensible formula, since there are several strange properties. The first is that our definition of Ricci curvature required that we input a vector (which we wrote, somewhat bizarrely, as two separate entries). In fact, the Riemannian metric g also requires that we input two vectors, so this is actually part of what makes the Ricci flow work.[43]

The second objection to considering this equation as a heat equation is that right hand side has a factor of -2 whereas the heat equation $\frac{\partial u}{\partial t} = \Delta u$ does not. This factor appears because the Ricci curvature should be thought of as the *negative* of a geometric Laplacian. In other words, the analytic Laplacian and geometric "Laplacian" differ by a sign.[44]

With these objections addressed, there is an apparent parallel between this formula and the heat equation $\frac{\partial u}{\partial t} = \Delta u$. We have tried to justify why it is possible to think of the Ricci curvature tensor as being analogous to a geometric Laplacian, which suggests that the Ricci flow is a heat flow of "shape". As a demonstration of this fact, it is worthwhile to see an example of the Ricci flow in action, which is depicted in Figure 22. There are also some animations of the flow, which are helpful to understand how it behaves [Pro]. In both the video and the figure, the surface becomes more spherical as time goes on. In the same way that the heat equation spreads the heat evenly throughout the space, the Ricci flow spreads the curvature evenly throughout the space.

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[43]Furthermore, both the Ricci curvature and the Riemannian metric are necessarily symmetric, which makes this formula sensible.

[44]It is important that the coefficient in front of the Ricci curvature is negative, or else the flow will not be defined for forward time (but instead for backwards time).

[45]To be more precise, this flow is converging to a round point, which means that it is shrinking to a point while asymptotically converging to a sphere.
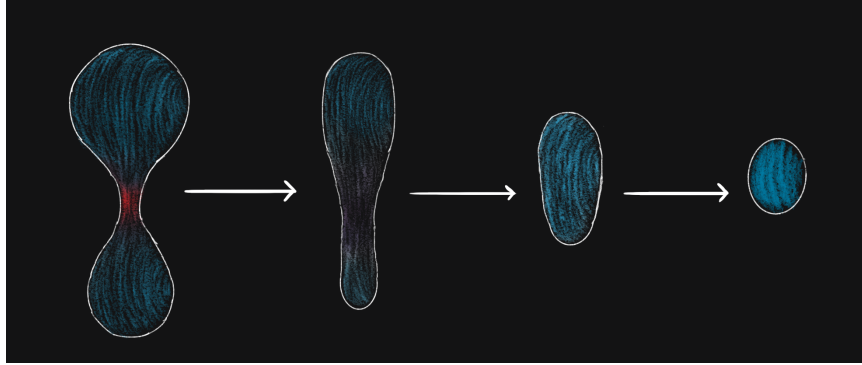
31

Figure 22: The Ricci flow converging to a round sphere[45]

Distinctions between the Ricci flow and heat equation

From the analogy between the heat equation and the Ricci flow, we might hope that the Ricci flow will smooth out our space and make it more uniform. In reality, the Ricci flow is more complicated than a heat flow, so this hope is too optimistic.

The first important distinction between the heat equation and the Ricci flow is that the latter is non-linear (linear combinations of solutions are no longer solutions) because curvature depends in a non-linear way on the metric. Second, it actually behaves more similarly to the reaction-diffusion equation

$$\frac{\partial}{\partial t}u = \Delta u + u^2. \tag{12}$$

The first term on the right-hand side behaves as a diffusion term that disperses heat throughout the space whereas the second acts a reaction term that concentrates heat at a point. Reaction-diffusion equations can be thought as a tug-of-war between the diffusion process and the reaction process. If diffusion wins, the solution will smooth itself out much like the normal heat equation. If the reaction term wins out, the heat will become more and more intense and can sometimes even become become infinite in
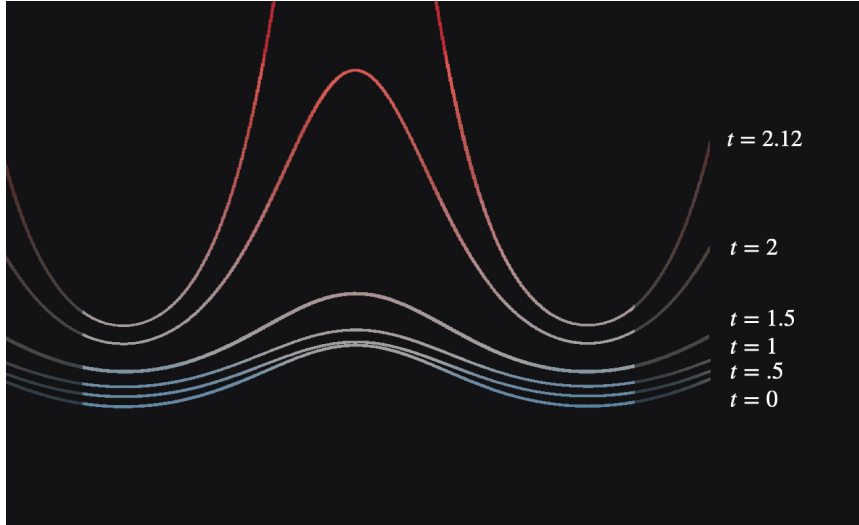
Figure 23: A reaction-diffusion equation going to infinity[47]

a finite amount of time.[46] An example of this is shown in Figure 23, where the solution becomes infinitely large after a short amount of time.

When we compute how the Riemann curvature (denoted R) of a space changes along Ricci flow, we find the following equation:

$$\frac{\partial}{\partial t}R = \Delta R + R^2 + R^\sharp. \tag{13}$$

It takes some background in Lie algebra to define the terms on the right-hand side of this formula [Wil13]. However, ignoring the final term on the right hand side, there is a notable similarity between this equation and Equation (12). In particular, sometimes the reaction term wins out and the curvature becomes larger and larger until the shape tears itself apart (or shrinks to a point).[48] However, understanding the formation of singularities is very

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[46]This occurs with the equation $\frac{\partial}{\partial t}u = \Delta u + u^2$ if the initial function is positive. For those who have taken a course on PDEs, it is a good exercise to show this using the maximum principle.

[47]Unlike with the standard heat equation, this equation does not have a closed form solution. As such, we used Fourier analysis to numerically approximate a solution.

[48]Once the short-time existence of the Ricci flow has been established, it is not hard to show the flow will exist until the sectional curvature goes to plus or minus infinity. As

difficult because it is not obvious when they occur or what they look like. One of Perelman's key contributions was to classify the possible structures of three-dimensional singularities and to show that a singularity known as the "cigar soliton" would not develop.
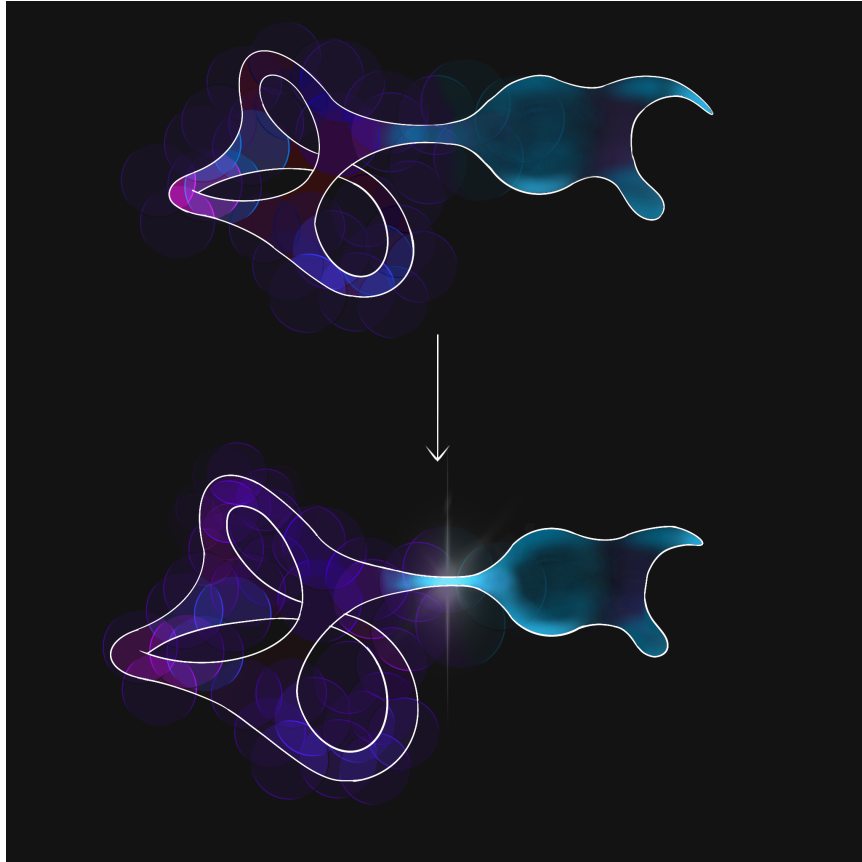

Figure 24: A Ricci flow becoming singular[49]

such, a singularity is generally defined to be a point in space-time where the norm of the Riemannian curvature tensor goes to infinity.

[49]Neck-like singularities do not occur for the Ricci flow on two-dimensional surfaces, so the space depicted here is three-dimensional. More precisely it is the connected sum of the unit tangent bundle for a hyperbolic surface (which has $\overline{\mathbb{SL}}_2$-geometry) and a deformed 3-sphere.

Recent progress and future directions

The Ricci flow is an active area of research, both as a tool to prove geometric and topological results and also as a topic of interest in its own right. It would be impossible to give a complete overview of the current state of research, but let us mention a few broad areas of interest.

The long-term behavior and singularity formation of the Ricci flow

The singularity formation and convergence of the Ricci flow is a fascinating and difficult topic. There are many open questions about when singularities occur and what their possible geometries can be. The most famous open problem in this direction is to determine whether the scalar curvature (i.e., the trace of the Ricci curvature) necessarily becomes infinite at a singularity.[50]

There has also been research into the structure of Ricci flows which are allowed to become singular and the singularities that form under "generic" initial conditions. For details on this line of work, we recommend the recent survey articles by Richard Bamler [Bam21] and Simon Brendle [Bre22].

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[50]Nataša Šešum showed that at any singularity, the Ricci curvature becomes infinite [Šeš05].

Geometric classification results

Apart from the Poincaré conjecture, the Ricci flow has been used to prove other geometric classification results. To give a few examples, Hamilton used Ricci flow to understand the geometry of three-dimensional spaces with positive Ricci curvature [Ham82] and four-dimensional spaces with positive curvature operator [Ham97]. In 2007, Brendle and Richard Schoen proved the Differentiable Sphere Theorem using the Ricci flow [Bre10] and Brendle recently used it to establish a partial classification of higher-dimensional spaces with positive isotropic curvature [Bre19].

There are also several ongoing programs which use Ricci flow (or some related flow) to attack open problems in geometry.

1. The *minimal model program* is a central focus of birational geometry, which is a branch of algebraic geometry involving rational functions. Jian Song and Gang Tian proposed using Kähler-Ricci flow (with surgery) to find the minimal models analytically. [ST17].

2. The Ricci flow tends to make the curvature of spaces more positive.[51] As such, it is very useful for studying and classifying spaces with positive curvature. For instance, the Ricci flow may be useful for fully classifying spaces which admit metrics of positive isotropic curvature or whose squared-distance has non-negative MTW tensor (see Chapter 12 of [Vil09] for a definition of this condition).

3. Finally, there are several problems concerning the geometry of four-dimensional spaces which seem amenable to a Ricci flow approach (see Section 5.6 of [Bam21] for some details).

Other geometric flows

The idea of using a heat-type flow to take geometric space and deform it to some canonical configuration predates the Ricci flow, but this approach

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[51]In other words, Ricci flow tends to preserve curvature positivity conditions (for details, see [BCRW19]). As a brief aside, there are a few negative curvature conditions that are preserved as well [KZ20].

has exploded in popularity in the four decades since the Ricci flow was first studied. At present, there are many geometric flows studied in the literature: mean curvature flow, harmonic map heat flow, Kähler-Ricci flow, Chern-Ricci flow, pluriclosed flow, anomaly flow, Calabi flow, Yamabe flow...

These flows play a significant role in differential geometry and have many applications, both within pure mathematics and more broadly. There is still much to be said about the behavior of geometric flows, and it is an active and vibrant area of research.

Endnotes

Further reading

Readers who are interested in the Poincaré conjecture may enjoy Donal O'Shea's book "The Poincaré Conjecture: In Search of the Shape of the Universe," which is intended for a general audience and gives a detailed history of the problem and its wider role in geometry [O'S08].

This paper discussed the uniformization theorem, but did not describe hyperbolic or spherical geometries in detail. For a basic introduction to this subject, I recommend the book "The Shape of Space" by Jeffrey Weeks. Chapter 6 of Tristan Needham's book "Visual Complex Analysis" provides an excellent description of the canonical two-dimensional geometries for those who are familiar with complex analysis (or are interested in learning about it) [Nee98]. Thurston's "The Geometry and Topology of Three-Manifolds" is excellent for learning about the three-dimensional geometries [Thu79].

For readers who are interested in learning differential geometry, I highly recommend Needham's recent book [Nee21] or John Lee's trilogy [Lee10, Lee13, Lee18]. For those conversant with differential geometry who are interested in geometric analysis, Schoen and Shing-Tung Yau's Lectures in Differential Geometry is excellent, although a more challenging read [SY94].

The discussion of curvature in this paper was strongly influenced by the synthetic theory of curvature bounds. I recommend the following paper of Villani for details on this approach [Vil16]. There were two reasons I chose this perspective. First, it makes it possible to discuss the geometric meaning

of curvature without needing to first define connections, parallel transport, etc. Second, the proof of the Geometrization conjecture uses Alexandrov geometry, so synthetic curvature plays a crucial role in the analysis of Ricci flow. The primary disadvantage of this approach is that it is practically impossible to use for computations, but this was not such an issue for an informal survey.

In order to study the Ricci flow, a good starting point is Peter Topping's manuscript "Lectures on Ricci flow" [Top06]. From there, Hamilton's paper on the formation of singularities is well worth reading [Ham93]. Understanding the full proof of the Poincaré conjecture is a massive undertaking (and the full Geometrization conjecture even more so), but there have been several surveys written and I recommend the following [KL08, MT07]. Furthermore, Danny Calegari recently wrote a chapter on the proof which emphasizes the three-dimensional geometry [Cal20].

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