DOES THE SAITO-KUROKAWA LIFT HAVE SOMETHING TO DO WITH T-DUALITY?

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Jun Zhang and I recently found a "mirror correspondence" between two spaces which appear in the literature on automorphic forms and algebraic geometry. At present, this duality is a bit mysterious to me and I'd like to know if there is some connection to number theory. The main purpose of this note is to pose several questions related to it. Most of the material is taken from Section 2 of [2], but I have condensed the exposition to focus on details which seem relevant to the number theoretic link.

To be concrete, I will focus on the complex two-dimensional case, which is the lowest dimension where the correspondence is non-trivial and where we have done most of the computations. However, all of this can be done in arbitrary dimension (i.e., arbitrary genus). As a disclaimer, I am a geometric analyst by background and have no expertise in algebraic geometry or number theory. As such, it is possible that these questions will be ill-posed.

1. Normal Distributions and their Information Geometry

In this section, we provide the information geometric motivation for studying the two spaces of interest. The construction of the spaces will be done in Section 2, so feel free to skip to that section if you are not interested in the statistics.

A univariate Gaussian distribution is a probability distribution of the form[1]

$$\rho(s : \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(s-\mu)^2}{2\sigma^2}\right).$$

In this expression, there are two parameters, μ and σ which correspond to the mean and variance of the distribution, respectively. We can consider the space of all univariate normal distributions as a *parametrized family*, which is a family of probability distributions specified by some number of parameters (in this case, the mean and variance).

Furthermore, we can consider the space of all univariate normal distributions as a *statistical manifold*, where the parameters serve as a global coordinate chart. For any parametrized family of probability distributions, it is possible to define an associated Riemannian metric, which is known as

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[1]Typically, normal distributions are written in terms of x instead of s. However, I will need x to mean something else in a moment, which is why I've made this notational switch.

the *Fisher metric*. Given an arbitrary parametrized family with parameters $\{\theta_i\}_{i=1}^n$ and sample space S, the Fisher metric is given by the expression

$$g\left(\frac{\partial}{\partial\theta_j}, \frac{\partial}{\partial\theta_k}\right) = \int_X \frac{\partial \log p(s,\theta)}{\partial\theta_j}\frac{\partial \log p(s,\theta)}{\partial\theta_k} p(x,\theta)\, ds,$$

This expression originates in statistics, and it can be interpreted as the infinitesimal form of the relative entropy. When we calculate the Fisher metric for normal distributions, we find the following,

$$g = \frac{1}{\sigma^2}(d\mu^2 + 2d\sigma^2).$$

As such, we have the following proposition, which was originally discovered by Shin-ichi Amari.

Proposition 1. *The moduli space of normal distributions is a hyperbolic half-plane (i.e., has constant negative curvature).*

1.1. **Normal distributions as an exponential family.** There are many reasons why normal distributions are important in statistics, but for our purposes the relevant property is that they form an *exponential family*, which is a parameterized family of probability distributions of a certain form.

Definition 1 (Exponential family)**.** *Given a sample space S, an exponential family is a parametrized family of probability distributions whose probability density functions are of the form*

$$\rho_S(s \,|\, x) = h(s) \exp\left(\sum_{i=1}^n x_i u^i(s) - \Phi(x)\right). \tag{1}$$

Here $h : S \to \mathbb{R}$ is a known function which serves to fix a base measure on S. The parameters are denoted by the x_i and take values in some domain $X \subset \mathbb{R}^n$. When an exponential family is parametrized in this way, the x_i are known as the natural parameters. The functions $u : S \to \mathbb{R}^n$ are known as the sufficient statistics. Finally, the function $\Phi : X \to \mathbb{R}$ is known as the log-partition function, which serves to renormalize the distribution so that the total mass is one.

In order to write normal distributions as an exponential family, we set

$$x_1 = \frac{\mu}{\sigma^2} \text{ and } x_2 = -\frac{1}{2\sigma^2}.$$

These functions are defined on the domain

$$\mathcal{X} = \{(x_1, x_2) \mid x_2 < 0\}.$$

We then take the sufficient statistics to be

$$u_1(s) = s, u_2(s) = s^2,$$

which are defined on the set

$$\mathcal{U} = \{(u_1, u_2) \mid u_1 \cdot u_1 - u_2 < 0\}.$$

Finally, the function $\Phi(x)$ is the following:

$$\Phi = -\frac{x_1 \cdot x_1}{4x_2} - \frac{1}{2}\log\left(\frac{x_2}{\pi}\right).$$

There is an important information geometric fact about exponential families which are crucial to the subsequent construction.

Proposition 2. *When parametrized in terms of the natural parameters x, the Fisher metric of an exponential family is given by the Hessian of the log-partition function Φ. That is to say, in the x-coordinates we have*

$$g_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}\Phi.$$

In this sense, the natural parameters form a set of "preferred coordinates" for the moduli space. There is also a *dual coordinate system*, which is induced by the sufficient statistics. More precisely, we have the following.

(1) Given an exponential family $\rho_S(s|\mathbf{x}_0)$, the expected value of the sufficient statistics

$$\overline{u}_i = \int_S u_i \rho_S(s|\mathbf{x}_0)\, ds$$

also form a coordinate chart for the exponential family.

(2) If we use these expected values as coordinates, the Fisher metric is given by the Hessian of the Legendre dual of the log-partition function Φ. In other words, we have

$$g\left(\frac{\partial}{\partial \overline{u}_i}, \frac{\partial}{\partial \overline{u}_j}\right) = \frac{\partial^2}{\partial \overline{u}_i \partial \overline{u}_j}\Phi^*,$$

where

$$\Phi^*(\overline{u}) = \inf_{x \in X}\langle x, \overline{u}\rangle - \Phi(x).$$

From here on out, I will drop the overline over $\overline{u}$ since there will be no ambiguity.

For the space of normal distributions, the Legendre dual of the log-partition function is

$$\Phi^* = -\frac{1}{2} - \log(u_2 - u_1 \cdot u_1).$$

This means that in the u-coordinates, the Fisher metric is given by

$$g = \frac{1}{(u_1 \cdot u_1 - u_2)^2}\begin{bmatrix} u_1 \cdot u_1 + u_2 & -u_1 \\ -u_1 & \frac{1}{2} \end{bmatrix}.$$

Since the natural parameters and expectation of the sufficient statistics can both be understood as different coordinates for the space of all normal distributions, the metric is always the same and has constant negative curvature.

2. Constructing the Statistical Mirrors

In order to construct Kähler manifolds from Hessian manifolds, we first recall the definition of a tube domain $T\Omega$.

Definition 2. *Given a domain $\Omega \subset \mathbb{R}^n$, its tube domain is the domain*

$$T\Omega = \{z \in \mathbb{C}^n \mid Re(z) \in \Omega, \, Im(z) \in \mathbb{R}^n\}.$$

These will be the primary spaces of interest throughout.

2.1. **The primal tube.** We first consider the half-space

$$\Omega = \{x_1, x_2 \mid x_2 < 0\}.$$

The tube domain $T\Omega$ is a half space in $\mathbb{C}^2$. Furthermore, we define a Kähler metric on this space using the potential function

$$\Psi(z_1, z_2) = -\frac{x_1 \cdot x_1}{4x_2} - \frac{1}{2}\log\left(\frac{x_2}{\pi}\right) \tag{2}$$

where $z_i = x_i + \sqrt{-1}y_i$. In other words, we set

$$\omega_{T\Omega} = \sum_{i,j} \sqrt{-1}\frac{\partial^2 \Psi}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j.$$

This space (with this particular metric) is known as the Siegel-Jacobi space, and has been studied in terms of modular forms. Its geometry is interesting, in that it is homogeneous and has constant scalar curvature, but is not symmetric. The geometry and arithmetic of this space was studied by Yang [4].

2.2. **The dual tube.** Note that the Kähler potential Ψ only depends on the x_i's. This allows us to restrict the function Ψ to Ω (instead of all of $T\Omega$). The potential Ψ, when restricted to Ω, is strictly convex, so we can compute its Legendre transform Ψ^*, which is defined to be

$$\Psi^*(u) = \sup_{x \in \Omega} \langle u, x \rangle - \Psi(x),$$

and is defined on the domain

$$\Omega^* = \left\{ u \in \mathbb{R}^n : \sup_{x \in \Omega} \langle u, x \rangle - \Psi(x) < \infty \right\}.$$

Doing so for the potential (2), we find that

$$\Psi^* = \frac{1}{2} - \frac{1}{2}\log(u_1 - u_2^2),$$

which is defined on the domain

$$\Omega^* = \{u_1, u_2 \mid u_1 - u_2^2 > 0\}.$$

From this, there is a second tube domain

$$T\Omega^* = \{w = u + \sqrt{-1}v \in \mathbb{C}^2 \mid u \in \Omega^*, v \in \mathbb{R}^2\}.$$

We can also define a Kähler metric using the function Ψ^*, extended to $T\Omega^*$ to be independent of v. Namely, we consider the Kähler form

$$\omega_{T\Omega^*} = \sum_{i,j} \sqrt{-1}\frac{\partial^2 \Psi^*}{\partial w^i \partial \bar{w}^j} dw^i \wedge d\bar{w}^j.$$

It turns out that $T\Omega^*$ is biholomorphically isometric to the disk, and thus has constant negative holomorphic sectional curvature (the induced metric also happens to be the Bergman metric). There is also a rich theory of automorphic forms on this space (see, e.g., [3]).

3. Remarks

(1) The correspondence between these two spaces is a slight modification of T-duality, which has been studied in context of mirror symmetry. However, neither of these spaces are Calabi-Yau (they aren't Ricci flat or compact) and there are no "singular fibers." As such, while it's perhaps accurate to call this a mirror correspondence, it's a very simple case of it. Furthermore, we are not quotienting the fibers by a lattice and its dual to have compact fibers.

(2) $T\Omega$ and $T\Omega^*$ are *not* biholomorphic. To see this, note that $T\Omega^*$ is biholomorphic to a unit ball in $\mathbb{C}^2$ whereas $T\Omega$ contains a complex line. As such, I think the relationship between these spaces is somewhat complicated.

(3) It is possible to obtain this duality in higher dimensions by considering the moduli space of normal distributions whose covariance matrix is ρId where ρ is a positive scalar.

(4) There are several papers in the literature which have studied the geometry of these spaces in depth. Both of these spaces have very interesting geometry, although it's not clear to me which properties are relevant for number theory and algebraic geometry.

4. Questions

These two spaces are interesting from a geometric point of view, but I'm curious about whether their duality has some applications in number theory or algebraic geometry. Here are some more specific questions about their geometry.

(1) Is this duality well known in number theory? If so, is there a standard reference?

(2) Is it possible to use the duality to say something about the geometry/arithmetic of these two spaces?

(3) Here is a more precise (and thus likely incorrect) version of the previous question.

The Saito–Kurokawa lift takes modular forms on the hyperbolic half plane $\mathbb{H}$ to a distinguished class (a *Spezialschar*) of Siegel modular forms on the Siegel half space. This lift can be understood as a

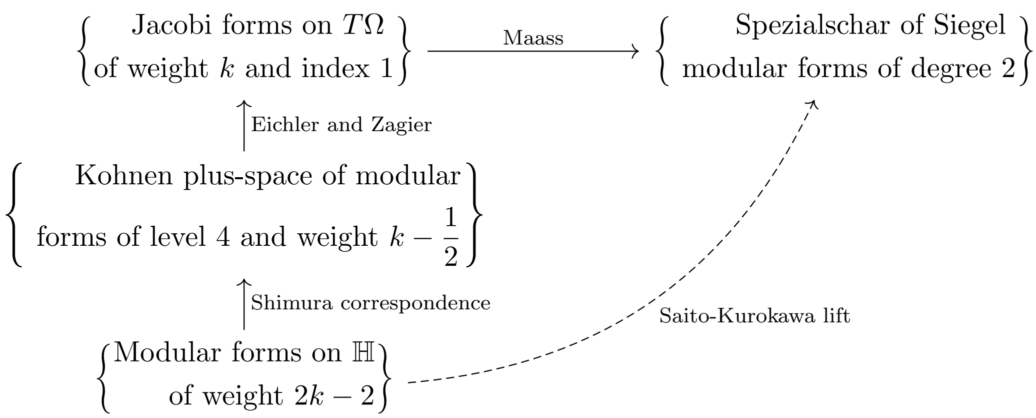

Figure 1. The Saito-Kurokawa lift

composition of three mappings, with the last map being a map from certain Jacobi forms (i.e., automorphic forms on $T\Omega$) to certain automorphic forms (i.e., a Spezialschar) on the Siegel upper half-space. This space naturally contains $T\Omega^*$ (as the space of symmetric complex matrices whose imaginary parts are positive definite and whose diagonal elements are equal) and may also have a similar interpretation as a "complexified exponential family." Keeping in mind that $T\Omega^*$ and $T\Omega$ are both models for the tangent bundle of the hyperbolic half plane, this picture suggests that there might be a natural way to understand the Saito-Kurokawa lift in terms of information geometry.

(4) If one considers the moduli space of isotropic multivariate normal distribution instead of univariate normal distributions, the associated statistical manifold is isometric to hyperbolic n-space and the dual Kähler metrics are the Siegel upper half space of degree n and the Siegel Jacobi space $\left(\mathbb{H}_{1,n}, ds^2_{1,n,1,1}\right)$ (using the notation of [4]). As a follow up question, is there any correspondence between the spectral theory of these spaces?

4.1. **The Moduli of Abelian varieties.** It is possible to interpret the Siegel half-space as the moduli space of principally polarized Abelian varieties (see Chapter 8 of [1]). As such, one can consider a point in $T\Omega^*$ as corresponding to the Abelian variety

$$T_M = \mathbb{C}^2 / \left(M\mathbb{Z}^2 + \mathbb{Z}^2\right)$$

where M is a complex matrix of the form

$$M = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_1 \end{bmatrix}$$

whose imaginary part is positive definite. Here, the diagonal elements are equal, so this is not a generic Abelian variety.

It is of interest whether there is some moduli space interpretation for the Siegel-Jacobi space $T\Omega$ as well. Naively, one may hope that $T\Omega$ is the moduli of a class of *dual* Abelian varieties with some extra condition, and that this duality provides the right notion of correspondence between these spaces.

Question 3. *Does $T\Omega$ have an interpretation as a moduli space of dual Abelian varieties?*

Furthermore, $T\Omega$ and $T\Omega^*$ are not merely complex varieties, but also Kähler manifolds, which means that they also have a natural notion of distance and curvature. It is also possible to write down Darboux coordinates for these spaces, so it is straightforward to compute volumes in these spaces.[2] As a result, it seems natural to ask whether these metrics are intrinsically meaningful for the moduli of Abelian varieties.

Question 4. *For genus g curves, the Weil-Petersson metric provides a natural metric on the moduli space, which provides insight into the geometry of such surfaces. Does the Kähler metric on $T\Omega^*$ provide an analogy of a "Weil-Peters son" for a class of principally polarized Abelian varieties? In other words, can we interpret it as a canonical Kähler metric in order to induce the "distance" between Abelian varieties? If so, is there a corresponding interpretation for the Kähler metric on $T\Omega$ in terms of dual Abelian varieties?*

References

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[2] Neither of these spaces have finite volume, so this does not give a natural way to discuss *random* Abelian varieties.