CURVE SHORTENING FLOW FOR CURVES IN $\mathbb{R}^n$

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Abstract. We follow Steven Altschuler's work on curves in $\mathbb{R}^3$ [2] and generalize the result to show that curves in Euclidean space of any dimension become asymptotically planar as they reach a singularity when deformed by curve shortening flow.

1. Foreword

This paper is mainly expository, following Altschuler's [2] work on curves in three dimensions while replacing the threes with n's but we introduce a conjecture and a heuristic argument in the final section as well. We aim for this to be an accessible and mostly self-contained exposition, introducing the work in a way that can be understood by an undergraduate interested in the subject. There are three main problems that we explore. The first is the proof that flow continues so long as the curvature is bounded, a result due to Altschuler and Grayson [5]. The second proof is to show that when a singularity occurs the curve becomes roughly planar. We follow the approach done by Altschuler but use generalized Frenet frames so that it holds in any dimension. Finally, we refer back to Altschuler's work and discuss the limiting the shape of the curve as they become singular.

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2. Introduction

Let γ be a smooth immersion from S^1 to R^n. We then define the following differential equation.

$$\partial_t \gamma = \kappa N \tag{1}$$

where κ is the curvature and N is the unit normal vector. We will study solutions to this equation, which consist of a family of curves γ_t with $t \in [0,\ \omega)$ which satisfy (1) with $\gamma_0 = \gamma$.

This is commonly referred to as curve shortening flow. When we compute the derivative of the length of the curve, we see that this flow is in some sense shortens the curve as quickly as possible in that it gives a gradiant flow for the length, which is where it gets its name. While there are not many examples that can be explicitly computed, there is one example that can be done easily. If one has a circle of radius R, the flow shrinks the circle homothetically (preserving the shape) to a point in time $\sqrt{R}$. A video showing curve shortening act on an a curve in the plane can be found at [4] which gives an excellent demonstration of the flow.

This flow has been studied extensively for 25 years, though most of the work has gone into curves in $\mathbb{R}^2$. The behavior of the curves in the plane is relatively well understood. Work by Richard Hamilton, Michael Gage and Matthew Grayson in the mid 1980's proved that embedded loops become convex and approach a circle before disappearing to a point. Non-embedded curves can have different limiting behavior; it is possible for a cusp to emerge or for the curve to approach a different shape as it converges to a point. Due to the work of Gerhard Huisken, Sigurd Angenent, Uwe Abresch, Joel Langer and others, the possible limiting shapes are well understood. Abresch and Langer were able to classify all of the curves whose behavior is similar to that of the circle in that they shrink homothetically [1] and that an arbitrary curve often approaches one of these curves as it deforms under curve shortening flow. In this paper we show that a curve in n-dimensional space becomes asymptotically planar as it shrinks. This was originally done for curves in $\mathbb{R}^3$ by Steven Altschuler. For higher dimensions, we can make modifications to his results so that they hold in n-dimensions as well. As a result, we call the final planarity result Altschuler's theorem as his method is the one used to prove the result.

As we begin our journey into this problem in geometric analysis, we are able to skip the two parts that are often most difficult for any partial differential equation, short-term existence and uniqueness of the solutions. Gage and Hamilton proved both for the most general case of the flow (a closed immersed submanifold of a Riemannian manifold flowing by its Laplacian) in their 1986 work. As part of their existence and uniqueness work, they proved that for any C^2 curve, for all time after zero that the flow is defined, the resulting curve is real analytic [7]. Therefore we will consider C^∞ curves, without loss of generality, and also refer to real analyticity occasionally. Rarely is it ever needed to complete an argument but it occasionally simplifies the proof. In two-dimensional curve-shortening flow, the techniques used to show that embedded curves approach a circle and shrink to a point are largely geometric. Spatial curves do not lend themselves to such geometric methods so easily and so the best way to attack the problem is by estimating the evolutions of the curvature and other geometric quantities. The natural quantities to study are those given by the Frenet frames.

In order to study curves in arbitrary dimensions, we use generalized Frenet frames in a way originally done by Camille Jordan [11]. In a sense that can be made precise, the curvature measures how quickly the curve deviates from a straight line and the torsion measures how quickly the curve deviates from its osculating plane. We can define other torsions using this same idea. This gives us a general frame and we can express the derivatives of the various vectors as such. By defining the vectors in the frame to point in the appropriate direction, we can choose the curvature and all of the torsions to be positive.

3. Notation

Here we review some of the notation used. To save space we wish to use the most concise possible notation for derivatives so we often denote

$$\frac{\partial^n}{\partial x^n} \text{ as } \partial_x^n.$$

If we wish to differentiate with respect to several variables we write

$$\frac{\partial^{n+m}}{\partial x^k \partial y^m} \text{ as } \partial_x^n \partial_y^m.$$

Since many of the partials do not commute, the order is important and so we have used this convention to make the order completely clear. We will use the following convention for the Frenet frame.

$$\partial_s \begin{pmatrix} T \\ N \\ B_1 \\ \vdots \\ B_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 & 0 & \cdots & 0 \\ -\kappa & 0 & \tau_1 & 0 & \cdots & 0 \\ 0 & -\tau_1 & 0 & \tau_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & -\tau_{n-2} & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ \vdots \\ B_{n-2} \end{pmatrix} \tag{2}$$

This resembles the three dimensional frame and throughout the proof the third and higher torsions do not play much of a role.

4. Geometric evolutions

We parametrize our curve by a time independent variable $u \in \mathbb{R}/2\pi$. This is useful because the derivatives of arclength and time do not commute so having a time-independent variable available allows for the initial calculations to be done.

We first calculate the derivative of velocity of the curve and from there, the way the partials with respect to time commute with the partials with respect to arclength. Since arclength and time are not independent variables, we cannot just switch the order of the partials. We do have a time independent variable and if we compute how the speed of the curve changes, then we can start to figure out how time and arclength are related to each other.

Let v be the velocity of the curve. Recall that

$$v^2 = \langle \partial_u \gamma, \partial_u \gamma \rangle. \tag{3}$$

We can take the temporal derivative of this expression with respect to time to get the following.

$$\begin{aligned} \partial_t v^2 &= 2\langle \partial_t \partial_u \gamma, \partial_u \gamma \rangle = 2\langle \partial_u \partial_t \gamma, \partial_u \gamma \rangle = 2\langle \partial_u \kappa N, \partial_u \gamma \rangle \\ &= 2\left\langle (\partial_u \kappa) N - v\kappa^2 T + v \cdot \kappa \cdot \tau_1 B_1, vT \right\rangle \\ &= -2v^2\kappa^2. \end{aligned}$$

However, $\partial_t v^2 = 2v \cdot \partial_t v = -2v^2\kappa^2$. So

$$\partial_t v = -\kappa^2 v. \tag{4}$$

Recalling also that $\partial_s = \partial_u / v$, we have the very first tool that we use for the proof. Using what we have just solved for, we can compute two things. The first is the partial commutator, which is extremely useful. The second is the derivative of the total length, which explains why the flow is called "curve-shortening."

Lemma 1. *Curve shortening flow shortens curves.*

Proof.

$$\partial_t \left(\int_0^{2\pi} v \; du \right) = \int_0^{2\pi} (\partial_t v) \; du = - \int_0^{2\pi} \kappa^2 v \; du = - \int_0^L \kappa^2 ds. \tag{5}$$

Therefore, the curve is always shrinking. Furthermore, since

$$\int_\gamma \kappa ds \geq 2\pi,$$

by Jensen's inequality we know that

$$-\int_\gamma \kappa^2 ds \leq \frac{-4\pi^2}{L} \leq \frac{-4\pi^2}{L(0)}$$

□

Since the length of the curve is strictly positive whenever the flow is defined, this shows that the flow is not defined for all time. Therefore there must be a singularity that prevents existence for all time. This is discussed in the next section.

As the arclength and time parameters are not independent, we do not expect their partials to commute. We calculate the commutator as follows:

$$\partial_t \partial_s = \partial_t \frac{1}{v} \partial u = \frac{1}{v} \partial_u \partial_t + \kappa^2 \frac{1}{v} \partial_u = \partial_s \partial_t + \kappa^2 \partial_s$$

Since we will repeatedly invoke this we number this equation.

$$\partial_t \partial_s = \partial_s \partial_t + \kappa^2 \partial_s \tag{6}$$

This allows us to move the temporal derivatives inward, which is very useful (a priori we only know how γ evolves with time so we need to move time derivatives inward). However, commuting comes at a cost of an extra term.

Now that we know how partials commute, the next step is to find how the curvature and first torsion evolve. We do this by calculating the evolutions of the tangent and normal vectors and then utilizing the Frenet frames to draw out the terms we want. We start with the curvature.

$$\partial_t \partial_s T = \partial_t(\kappa N) = (\partial_t \kappa) N + \kappa \partial_t N. \tag{7}$$

This equation helps us because we know

$$\langle N, N \rangle = 1 \Rightarrow \partial_t \langle N, N \rangle = 2 \langle \partial_t N, N \rangle = 0$$

and therefore

$$\langle \partial_t \partial_s T, N \rangle = \partial_t \kappa. \tag{8}$$

This calculation tells us that the temporal derivative of curvature is the coefficient of N when we calculate $\partial_t \partial_s T$. So

$$\partial_t \partial_s T = \partial_s \partial_t T + \kappa^2 \partial_s T = \partial_s(\partial_t T) + \kappa^3 N. \tag{9}$$

Using the commutator and the definition of the tangent vector, we obtain

$$\begin{aligned} \partial_t T &= \partial_t \partial_s \gamma = \partial_s \partial_t \gamma + \kappa^2 \partial_s \gamma = \partial_s(kN) + \kappa^2 T \\ &= (\partial_s K) N + \kappa \tau_1 B_1 - \kappa^2 T + \kappa^2 T = \partial_s \kappa N + \kappa \tau_1 B_1. \end{aligned}$$

Plugging back into the original equation we find

$$\begin{aligned}\partial_t\partial_s T &= (\partial_s^2\kappa)N - \kappa\partial_s kT + \tau_1(\partial_s\kappa)B_1 + (\partial_s\kappa)\tau_1 B_1\\ &\quad +\kappa\partial_s\tau_1 B_1 - \kappa\tau_1^2 N + \kappa^3 N - \kappa\tau_1^2 N + \kappa\tau_1\tau_2 B_2,\end{aligned}$$

and so finally

$$\partial_t\kappa = \partial_s^2\kappa + \kappa^3 - \kappa\tau_1^2. \tag{10}$$

The evolution of first torsion is very messy but it can be calculated using the same idea. We compute $\partial_t N$ next.

$$\partial_t N = -(\partial_s\kappa)T + \frac{2\tau_1}{\kappa}(\partial_s\kappa)B_1 + (\partial_s\tau_1)B_1 + \tau_1\tau_2 B_2. \tag{11}$$

This is obtained by considering all the terms not in the normal direction when we calculated $\partial_t\kappa N$ and then dividing by κ. From here we know that

$$\partial_t\partial_s N = -(\partial_t\kappa)T - (\partial_t T) + (\partial_t\tau_1)B_1 + \tau_1(\partial_t B_1). \tag{12}$$

So we need to find the appropriate part of the coefficient in front of B_1. There is a part of $\partial_t T$ in the first binormal direction as well so we must disregard this part. By the partial commutator, we know

$$\partial_t\partial_s N = \kappa^2\partial_s N + \partial_s\partial_t N. \tag{13}$$

Substituting in the RHS of (12) into the above, we have that

$$\partial_t\partial_s N = \kappa^2\partial_s N + \partial_s\left(-(\partial_s\kappa)T + \frac{2\tau_1}{\kappa}\partial_s\kappa B_1 + \partial_s\tau_1 B_1\right). \tag{14}$$

Now we just need to wade through the algebra. The above equation becomes.

$$\begin{aligned}\partial_t\partial_s N &= \kappa^2\partial_s N - (\partial_s^2\kappa)T - \partial_s k\cdot\kappa N + \partial_s\left(\frac{2\tau}{\kappa}\frac{\partial\kappa}{\partial s}\right)B_1\\ &\quad + \left(\frac{2\tau_1}{\kappa}\partial_s\kappa\right)(-\tau_1 N + \tau_2 B_2) + \partial_s^2\tau B_1 + \frac{\partial\tau_1}{\partial s}(-\tau_1 N + \tau_2 B_2).\end{aligned}$$

This simplifies to

$$\begin{aligned}\partial_t\partial_s N &= -\kappa^3 T - \partial_s^2 T - (\partial_s\kappa)\kappa N + \left(\partial_s\left(\frac{2\tau_1}{\kappa}\frac{\partial\kappa}{\partial s}\right) + \frac{\partial^2\tau_1}{\partial s^2}\right)B_1\\ &\quad +\kappa^2\tau_1 B_1 + \frac{2\tau_1\tau_2}{\kappa}(\partial_s\kappa)B_2 + \frac{\partial\tau_1}{\partial s}\tau_2 B_2.\end{aligned}$$

Now we only must consider the terms in front of B_1 that do not appear as terms in $-\kappa(\partial_t T)$. Doing this, we finally obtain

$$\partial_t \tau_1 = 2\kappa^2 \tau_1 + \partial_s \left(\frac{2\tau_1}{\kappa} \partial_s \kappa \right) + \partial_s^2 \tau_1 - \tau_1^2 \tau_2. \tag{15}$$

Much later it is helpful to write this equation in the following form:

$$\partial_t \tau_1 = \partial_s^2 \tau_1 + 2\frac{1}{\kappa}(\partial_s \kappa)(\partial_s \tau_1) + \frac{2\tau_1}{\kappa} \left(\partial_s^2 \kappa - \frac{1}{\kappa}(\partial_s \kappa)^2 + \kappa^3 \right) - \tau_1^2 \tau_2. \tag{16}$$

In theory, one could compute time derivatives of any quantity in the Frenet frame. In practice this would be very tiresome and so for now we are done with the frame. We return to it later when we have some more tools. Working in the frame has given us three equations ((6), (10), and (16)) which are very useful.

5. Singularities

In the simplest possible terms, a singularity is a time and location on the curve for which we can no longer continue to deform the curve according to the equation. In the following section we prove that the curve remains smooth so long as the curvature remains bounded so the singularities occur ony when the curvature blows up. As shown in Lemma 1, a singularity must always occur since the length of the curve is strictly positive and there is a strictly negative upper bound on the temporal derivative of the length of the curve. Therefore, studying the behavior of the equation comes down to analyzing the possible singularities. We would therefore like to know exactly when these singularities occur and it turns out the flow continues so long κ is finite everywhere on the curve. This is proved in the following section.

To study the singularity, we pick a sequence of points on the curve and times (p_m, t_m) such that t_m converges to some time ω and the curvature at (p_m, t_m) goes to infinity. We call this a blow-up sequence. Note that the time ω at which a blow-up sequence occurs is unique because after this time the flow is no longer defined. Even if there were a way to extend the equation past this time, we do not do so. The uniqueness of singular times is actually needed at one point later on. Most of the bounds that we obtain are in terms of the maximum of curvature at a certain time and if we cannot bound the ratio of the maximum curvature to the curvature along the blow-up sequence then we would run into problems. Therefore, we add a technical condition to a blow-up sequence and if a sequence satisfies this condition, we call it an essential blow-up sequence. If we define

$$M_t = \sup_{p\in\gamma} \kappa^2(p,t), \tag{17}$$

then we do not want our curvature squared along our sequence to be too much smaller than this value. Therefore we define an essential blow-up sequence as a blow-up sequence where $\kappa^2(p_m,t_m) \geq \rho M_{t_m}$ for some $\rho > 0$ and all $m \in \mathbb{N}$. An essential blow-up sequence always exists and for many curves, all blow-up sequences are essential. A curve is planar if it can be embedded in the plane. For real analytic curves, this is equivalent to $\tau_1 = 0$ everywhere on the curve (except inflection points, where it is not defined). We call a singularity is roughly planar if

$$\lim_{m\to\infty} \frac{\tau_1}{\kappa} = 0$$

along an essential blow-up sequence (p_m,t_m). Now that we have defined these terms, we can properly state the theorems that we wish to prove.

Theorem 2. *Given γ_t a solution to* (1) *and an essential blow-up sequence (p_m,t_m) on γ_t,*

$$\lim_{m\to\infty} \left(\frac{\tau_1}{\kappa}(p_m,t_m)\right)^2 = 0$$

(i.e. the singularity is roughly planar)

6. Longer Term Existence

Although we know that given a smooth curve, we can always allow curve shortening to flow for a short amount of time by [7], we do know that some sort of singularity must arise so this flow cannot occur for all time. We claim that the flow continues so long as curvature remains bounded. This proof is originally due to Steven Altschuler and Matthew Grayson in [5]. We do this by assuming curvature is bounded, and then bounding $||\partial_s^n T||^2$. We do this by calculating $\partial_t\partial_s^n T$.

$$\text{Let } M^n_{t_0} = \sup_{t=t_0} ||\partial^n_s T||^2. \tag{18}$$

Notice that $M^1_{t_0} = M_{t_0}$ as defined before. We inductively show that

$$\partial_t M^n_{t_0} \leq A M^n_{t_0} + B\sqrt{M^n_{t_0}} + C, \tag{19}$$

with A, B and C constants. Integrating this inequality, we obtain that $M^n_{t_0}$ is bounded above by an exponential function and so is finite so long as curvature is finite.

6.1. **Maximum Principle.** There is a maximum principle that is very useful for this proof. Since our curve is compact at any time, it achieves its maximum value at some point p on the curve (we could have defined $M^n_{t_0}$ in terms of maximums instead of supremums because of this). At this point

$$\partial^2_s ||\partial^n_s T||^2 \leq 0$$

or else p wouldn't be a maximum. Therefore, if we take the derivative and find such a term, we can disregard when considering the upper bound of $||\partial^n_s T||^2$. This is very useful because it provides a condition on which we can disregard certain higher derivatives which is normally very difficult to do. This idea is often used and is a special case of a much more general principle of partial differential equations. This version is sufficient for our purposes.

6.2. **Initial Bounding Results.** We first consider $\partial_t \partial^n_s T$. Repeatedly using the partial commutation rule and recalling $\kappa^2 = ||\partial_s T||^2$, we obtain

$$\partial_t \partial^n_s T = \partial^n_s \partial_t T + \sum_{i=0}^{n-1} \partial^i_s \left(||\partial_s T||^2 \partial^{n-i}_s T \right).$$

Remembering that $T = \partial_s \gamma$, we have

$$\partial^n_s (\partial_t \partial_s \gamma) + \sum_{i=0}^{n-1} \partial^i_s \left(|\partial_s T|^2 \partial^{n+1-i}_s \gamma \right).$$

Commuting the derivative once more, we obtain

$$\partial_t \partial^n_s T = \partial^{n+2}_s T + \sum_{i=0}^{n} \partial^i_s \left(|\partial_s T|^2 \partial^{n-i}_s T \right).$$

Now, $\partial_t||\partial_s^n T||^2 = 2\langle \partial_t \partial_s^n T, \partial_s^n T\rangle$. Substituting in, we find

$$\partial_t||\partial_s^n T||^2 = 2\langle \partial_s^{n+2}T + \sum_{i=0}^{n} \partial_s^i \left(||\partial_s T||^2 \partial_s^{n-i} T\right), \partial_s^n T\rangle$$

We wish to only consider the terms that contain a derivative of order n or greater. The other terms can be bounded inductively. Therefore, we obtain

$$\begin{aligned}
\partial_t||\partial_s^n T||^2 &= 2\left\langle \partial_s^{n+2}T + (n+1)\partial_s^n T \cdot ||\partial_s T||^2 + \right. \\
&\quad 2\langle \partial_s^{n+1}T, \partial_s T\rangle T + 2\langle \partial_s^n T, \partial_s^2 T\rangle T \\
&\quad + \ 2(n+1)\langle \partial_s^n T, \partial_s T\rangle \partial_s T, \partial_s^n T\rangle \\
&\quad + \left\langle \sum \text{ Lower Order Terms }, \partial_s^n T \right\rangle
\end{aligned}$$

The second term comes from differentiating $\partial_s^{n-i}T$ i more times. The third comes from differentiating $||\partial_s T||^2$ exactly n times. The fourth and fifth come from differentiating $||\partial_s T||^2$ exactly $n-1$ times.

We can manipulate the first term in the following way:

$$\begin{aligned}
\partial_s^2||\partial_s^n T||^2 &= \partial_s^2\langle \partial_s^n T, \partial_s^n T\rangle \\
&= 2\langle \partial_s^{n+2}T, \partial_s^n T\rangle + 2\langle \partial_s^{n+1}T, \partial_s^{n+1}T\rangle.
\end{aligned}$$

Substituting into the equation above, we find

$$\begin{aligned}
\partial_t||\partial_s^n T||^2 &= \partial_s^2||\partial_s^n T||^2 - 2\langle \partial_s^{n+1}T, \partial_s^{n+1}T\rangle + 2\langle (n+1)\partial_s^n T \cdot ||\partial_s T||^2 \\
&\quad +2\langle \partial_s^{n+1}T, \partial_s T\rangle T + 2\langle \partial_s^n T, \partial_s^2 T\rangle T \\
&\quad +2(n+1)\langle \partial_s^n T, \partial_s T\rangle \partial_s T, \partial_s^n T\rangle \\
&\quad +\langle \sum \text{ Lower Order Terms }, \partial_s^n T\rangle.
\end{aligned}$$

Recalling that the norm of an inner product is less than the product of the norms and completing the square, we obtain

$$\begin{aligned}
\partial_t||\partial_s^n T||^2 &\leq \partial_s^2||\partial_s^n T||^2 - 2\left(||\partial_s^{n+1}T|| - ||\partial_s T|| \cdot ||\partial_s^n T||\right)^2 + \\
&\quad 2(n+2)||\partial_s T||^2||\partial_s^n T||^2 + 4n||\partial_s^n T||^2||\partial_s^2 T|| + \\
&\quad 4(n+1)||\partial_s T|| \cdot ||\partial_s^n T||^2 + \\
&\quad \left(\sum \text{ Lower Order Terms}\right) \cdot ||\partial_s^n T||
\end{aligned}$$

With all of this at hand, we can finally prove that $\partial_t||\partial_s^n T||^2$ is bounded whenever curvature is bounded so the flow gives a smooth curve so long as curvature does not blow up.

Lemma 3. *Curve shortening flow* (1) *does not become singular for a curve* γ *where* $\sup_{p\in\gamma}\kappa^2$ *is bounded.*

Proof.

$$\begin{aligned}\partial_t(M_t^n) \quad &\leq \quad \partial_s^2 M_t^n + 2(n+2)M_t^1 \cdot M_t^n \\ &\quad +4n\sqrt{M_t^2}\cdot M_t^n + 4(n+1)\sqrt{M_t^1}\cdot M_t^n \\ &\quad + \left(\sum \text{ Lower Order Terms}\right)\cdot\sqrt{M_t^n}\end{aligned}$$

Suppose $\kappa(p,t)^2 \leq M_t \leq M$ for $t\in[0,\alpha]$. As mentioned before $\partial_s^2 M_t^n \leq 0$ so this term can be disregarded. If we can bound M_t^k for all $k<n$, then

$$\partial_t(M_t^n) \leq A\cdot M_t^n + B\cdot\sqrt{M_t^n} + C,$$

whereA, B, and C are real numbers. We can bound the solution of this differential equation by an exponential function and so M_t^n is bounded as well. To finish this as an inductive proof, we must prove the base case and show that M_t^2 is bounded. Plugging into the equation, we find

$$\begin{aligned}\partial_t||\partial_s^2 T||^2 \quad &= \quad \partial_s^2||\partial_s^2 T||^2 - 2\langle\partial_s^3 T,\partial_s^3 T\rangle + 2\langle 3\partial_s^2 T\cdot||\partial_s T||^2 \\ &\quad +2\langle\partial_s^3 T,\partial_s T\rangle T + 2\langle\partial_s^2 T,\partial_s^2 T\rangle T + 6\langle\partial_s^2 T,\partial_s T\rangle\partial_s T,\partial_s^2 T\rangle \\ &\quad +\langle\sum \text{ Lower Order Terms },\partial_s^2 T\rangle.\end{aligned}$$

Recalling that $\langle T,\partial_s^2 T\rangle = ||\partial_s T||^2$ from the Frenet frames, we can simplify this to

$$\begin{aligned}\partial_t||\partial_s^2 T||^2 \quad &\leq \quad \partial_s^2||\partial_s^2 T||^2 - 2\left(||\partial_s^3 T|| - ||\partial_s T||\cdot||\partial_s^2 T||\right)^2 \\ &\quad +8||\partial_s T||^2||\partial_s^2 T||^2 + 8||\partial_s^2 T||^2||\partial_s T||^2 + 12||\partial_s T||\cdot||\partial_s^2 T||^2 \\ &\quad + \left(\sum \text{ Lower Order Terms}\right)\cdot\sqrt{M_t^n}.\end{aligned}$$

Repeating the same argument from before, we obtain an exponential bound on $||\partial_s^2 T||^2$. By induction, $\partial_t||\partial_s^n T||^2$ and $||\partial_s^n T||^2$ are bounded for all

n whenever curvature is bounded and so the curve remains smooth and the flow continues. □

7. Altschuler Bounds

The previous bounds were able to solve the existence problem but do not give us more information about the behavior of the flow. We must produce stronger bounds due to Altschuler in [2]. These stronger bounds and the corollary following them are the heart of the theorem from which everything else can be produced.

Lemma 4. *On the interval*

$$\left[t_m, t_m + \frac{1}{8M_{t_m}}\right],$$

we have that

$$||\partial_s^n T||^2 \leq \frac{c_n M_{t_m}}{(t-t_m)^{n-1}}. \tag{20}$$

Proof. We use another inductive proof. We start with $||\partial_2 T||^2$. Plugging into the relevant equation, we find

$$\partial_t||\partial_s^2 T||^2 = \partial_s^2||\partial_s T||^2 - 2||\partial_s^2 T||^2 + 4||\partial_s T|^4 + \left(2\cdot\partial_s||\partial_s T||^2\right)\cdot\langle T, \partial_s T\rangle.$$

However, the last term is zero since the tangent vector is a unit vector. Therefore,

$$\partial_t||\partial_2 T||^2 \leq \partial_s^2||\partial_s T||^2 + 4||\partial_s T||^4$$

.

By the same maximum principle used before, at the point where curvature is maximized, $\partial_s^2||\partial_s T||^2 \leq 0$ and so

$$\partial_t M_t \leq 4\cdot(M_t)^2.$$

Now this differential equation we can explicitly solve by separation of variables and then integration. Upon doing so, we obtain that

$$-\frac{1}{M_t} + \frac{1}{M_0} \leq 4t.$$

Therefore, if $t \leq 1/(8\cdot M_0)$, then

$$M_t \leq 2\cdot M_0.$$

The fact that we are starting from $t = 0$ in this proof is of no consequence because the flow is not time dependent and so any solution can be translated in time to this starting time.

Altschuler now has a clever idea, which is to try to establish

$$\partial_t \left(\sum_{i=0}^{n} a_i \; t^{i-1} ||\partial_s^i T||^2 \right) \le A_n \cdot M_0 \text{ for some } A_n, a_i \in \mathbb{R} \tag{21}$$

The reason to do this is that $\partial_t ||\partial_s^n T||^2$ contains the term $-2||\partial_s^{n+1} T||^2$ so we expect that if make the a_i terms large enough we can eliminate most of the terms of $\partial_t ||\partial_s^{n+1} T||^2$. Using this idea,

$$\begin{aligned}
\partial_t \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) \;\; \le \;\; & \partial_s^2 \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) + \; ||\partial_s^2 T||^2 \\
& -2t \left(||\partial_s^3 T|| - ||\partial_s T|| \cdot ||\partial_s^2 T|| \right)^2 \\
& 16t \cdot ||\partial_s T||^2 ||\partial_s^2 T||^2 - 8||\partial_s^2 T||^2 + 16||\partial_s T||^4
\end{aligned}$$

But $||\partial_s T||^2 \le 2 \cdot M_0$ on this interval so

$$\begin{aligned}
\partial_t \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) \;\; \le \;\; & \partial_s^2 \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) \\
& + (32 M_0 t - 7) \cdot ||\partial_s^2 T||^2 + 64 (M_0)^2
\end{aligned}$$

However, since

$$t < \frac{1}{8 \cdot M_0} \text{ and } (32 M_0 t - 7) < 0,$$

it follows

$$\partial_t \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) \le \partial_s^2 \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) + 64 M_0^2.$$

At the point that maximizes this function,

$$\partial_s^2 \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) \le 0$$

and so we can disregard the first term. Therefore, at the supremum of the function

$$\partial_t \left(t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \right) \le 64 M_0^2.$$

If we now integrate out this differential equation, we find that

$$t \cdot ||\partial_s^2 T||^2 + 4||\partial_s T||^2 \leq 4M_0 + 64M_0^2 t \leq 12M_0,$$

and finally

$$||\partial_s^2 T||^2 \leq \frac{12M_0}{t}.$$

We can continue the induction from here, although we will not fill in all of the details as the algebra quickly becomes unmanageable.

$$\begin{aligned}
\partial_t \left(t^{n-1}||\partial_s^n T||^2\right) \quad \leq \quad & t^{n-1}\left(\partial_s^2 ||\partial_s^n T||^2 - 2(||\partial_s^{n+1}T|| - ||\partial_s T|| \cdot ||\partial_s^n T||\right)^2 \\
& +2(n+2)||\partial_s T||^2||\partial_s^n T||^2 + 4n||\partial_s^n T||^2||\partial_s^2 T|| \\
& +4(n+1)||\partial_s T|| \cdot ||\partial_s^n T||^2 \\
& + \left(\sum \text{ Lower Order Terms}\right) \cdot ||\partial_s^n T||) \\
& +(n-1)t^{n-2}||\partial_s^n T||^2.
\end{aligned}$$

Now, either

$$2(||\partial_s^{n+1}T|| - ||\partial_s T|| \cdot ||\partial_s^n T||)^2 \geq \frac{1}{2}||\partial_s^{n+1}T||^2$$

$$\text{or } ||\partial_s T|| \cdot ||\partial_s^n T|| \geq \frac{1}{2}||\partial_s^{n+1}T||.$$

In the former case, we can repeatedly use the use the Peter Paul inequality to complete the proof and in the latter case it follows immediately since

$$||\partial_s T|| \cdot ||\partial_s^n T|| \leq 2M_t \cdot \frac{a_n M_t}{t^{n-1}} \text{ and } \leq \frac{1}{8M_t}.$$

□

7.1. **Time Translation.** These are useful but do not help us near time t_m. Altschuler produces another clever idea that allows us to turn these bounds into something much more useful. The idea is to pick ρ such that $\rho * M_t < M_{t_n}$ for $t < t_n$ and use an earlier time so that the bounds can be expressed in terms of M_{t_m}. We can pick a ρ that satifies this condition because if there were some sequence of points $(\hat{p}_k, \hat{t}_k)$ such that along some subsequence (p_{m_k}, t_{m_k}) of the essential blow-up sequence such that

$$\frac{\kappa(\hat{p}_k, \hat{t}_k)}{\kappa(p_{m_k}, t_{m_k})} = \infty,$$

then there is another blow-up sequence at $\lim_{k\to\infty} \hat{t}_k = \hat{\omega}$. But $\hat{\omega} \neq \omega$ or else the blow-up sequence would not be essential. However, blow-up times are unique so this cannot happen (this is where uniqueness of blow-up times is needed). By the continuity of M_t, given a blow-up sequence (p_m, t_m), there exists another sequence $(\tilde{p}_m, \tilde{t}_m)$ such that

$$t_n = \tilde{t}_m + \frac{1}{32M_{\tilde{t}_m}}.$$

Then, by the previous argument, we know that

$$\frac{M_{t_m}}{\rho} > M_{\tilde{t}_m},$$

and so

$$t_n + \frac{3\rho}{64M_{t_m}} = \tilde{t}_m + \frac{1}{32M_{\tilde{t}_m}} + \frac{3\rho}{64M_{t_m}} < \tilde{t}_m + \frac{1}{8M_{\tilde{t}_m}}.$$

Therefore,

$$\left[t_m, t_n + \frac{3\rho}{64M_{t_m}}\right] \subset \left[\tilde{t}_m, \tilde{t}_m + \frac{1}{8M_{\tilde{t}_m}}\right].$$

Plugging this into the Altschuler bounds,

$$||\partial_s^n T||^2 \leq \frac{\tilde{c}_n M_{\tilde{t}_m}}{(t - \tilde{t}_m)^{n-1}}$$

which implies,

$$||\partial_s^n T||^2 \leq \frac{\tilde{c}_n M_{t_m}}{\rho\left(\frac{1}{32M_{\tilde{t}_m}}\right)^{n-1}} = \frac{\tilde{c}_n M_{t_m}}{\rho} \cdot \left(32M_{\tilde{t}_m}\right)^{n-1} < \tilde{c}_n \left(\frac{32M_{t_m}}{\rho}\right)^n$$

for all

$$t \in \left[t_m, t_n + \frac{3\rho}{64M_{t_m}}\right].$$

This proves the following bounds.

Lemma 5. *Given an essential blow-up sequence (p_m, t_m), there exists $\rho > 0$ such that for all*

$$t \in \left[t_m, t_m + \frac{3\rho}{64M_{t_m}}\right],$$

$$||\partial_s^n T||^2 \leq c_n (M_{t_m})^n. \tag{22}$$

This result is at the heart of the planarity theorem. It states that on a certain time interval near an essential sequence, we can bound $||\partial_s^n T||^2$ by particularly nice quantities related to κ^2. The exact size of this time interval is not particularly important, just that the interval is large enough for certain estimates to go through. We often use this result in the form of the following corollary, which states the result in terms of the Frenet frame.

Corollary 6. *On the time interval* $I = \left[t_m, t_n + \frac{3\rho}{64 M_{t_m}}\right]$,

$$\begin{aligned}
c_1 M_{t_m} &\geq \kappa^2 \\
c_2 (M_{t_m})^2 &\geq (\partial_s \kappa)^2 \\
c_2 (M_{t_m})^2 &\geq (\kappa \tau_1)^2 \\
c_3 (M_{t_m})^3 &\geq (\partial_s^2 \kappa - \kappa^3 - \kappa \tau_1^2)^2 \\
c_3 (M_{t_m})^3 &\geq (2\tau_1 \partial_s \kappa + \kappa \partial_s \tau_1)^2 \\
c_4 (M_{t_m})^4 &\geq (\kappa \partial_s^2 \tau_1 + 3 \partial_s \kappa \cdot \partial_s \tau_1 + 3\tau_1 \partial_s^2 \kappa \\
&\quad -\kappa^3 \tau_1 - \kappa \tau_1^3 - \kappa \tau_1 \tau_2^2)^2.
\end{aligned}$$

Proof. We start by differentiating the tangent vector in terms of the Frenet frame four times.

$$\begin{aligned}
\partial_s T &= \kappa \cdot N \\
\partial_s^2 T &= -\kappa^2 \cdot T + \partial_s \kappa \cdot N + \kappa \tau_1 \cdot B_1 \\
\partial_s^3 T &= -3\kappa \partial_s \kappa \cdot T + (\partial_s^2 \kappa - \kappa^3 - \kappa \tau_1^2) \cdot N \\
&\quad + (2\tau_1 \partial_s \kappa + \kappa \partial_s \tau_1) \cdot B_1 + \kappa \tau_1 \tau_2 \cdot B_2 \\
\partial_s^4 T &= (\kappa\ \partial_s^2 \tau_1 + 3 \partial_s \kappa \cdot \partial_s \tau_1 \\
&\quad + 3\tau_1 \partial_s^2 \kappa - \kappa^3 \tau_1 - \kappa \tau_1^3 - \kappa \tau_1 \tau_2^2) B_1 + \ldots
\end{aligned}$$

Applying the Bounding Lemma, we obtain the corollary. Note that the c_n constants are the same as in the previous lemma and so depend only on ρ. $\square$

This gives us bounds in terms of more useful geometric quantities. While it may seem arbitrary to stop at the fourth derivative and only consider certain terms, it turns out that these terms are the ones that are needed to continue with the proof. Notice that with sufficiently bad vision, the fourth term looks a lot like $\partial_t \kappa$ where as final term looks like $\kappa \cdot \partial_t \tau_1$. This observation is crucial and explored in the next section.

8. Dissipation Estimates

We wish to show that if we have an essential blow-up sequence, there is a neighborhood in both space and time in which the curvature remains large. This motivates the following definition:

Definition. *Given $d > 0$ and $(p_m, t_m) \in S^1 \times [0, \omega)$, we define the neighbourhood $\mathbf{N}(p_m, t_m, d)$ to be the following set*

$$\mathbf{N}(p_m, t_m, d) = \left\{ (p,t) \in S^1 \times [t_m, \omega) : \, dist_{t_m}(p_m, p) \leq \sqrt{\frac{d}{M_{t_m}}} \,, t - t_n \leq \frac{d}{M_{t_m}} \right\}.$$

Notice that the distances with respect to arclength only depend on the time t_m and is invariant in the t component of (p,t).

Although it may seem arbitrary, there are several reasons to define this neighborhood as above. Firstly, as curvature varies with spatial dilation squared, the dimensions satisfy dimensional analysis. Secondly, if one were to integrate κ^3 along an essential blow-up sequence in these neighborhoods, one would expect that you could bound this quantity from below as the size of the term is roughly inversely proportional to the size of the region you are integrating over. We will bound this integral away from zero from below and show that a similar integral regard τ_1 goes to zero and use this to show the rough planarity. As always, there are technical details and we address them now.

We must make sure that curvature stays large enough everywhere in the neighborhood. We show that the curvature cannot dissipate too quickly and that with the appropriate radius chosen the maximum value of curvature in a neighborhood is no more that twice the minimum, independent of which term in the blow-up sequence we consider. We must then do the same thing for torsion, which is the only place where any difficulty in generalizing the proof from three to higher dimensions might arise. The reasons to do this for torsion are so that once we show that

$$\int_{\mathbf{N}(p_m,t_m,d)} \tau_1^2\ \kappa\ ds\ dt \to 0 \text{ as } m \to \infty,$$

we can conclude the same thing for the integrand point-wise (This computation is done in the next section).

Our general strategy is to bound the temporal and spatial derivatives Altschuler's growth bounds. Then, by using a first-order Taylor approximation, we can prevent the quantities from dissipating too quickly in space or forward time.

8.1. **Curvature Dissipation.** $\exists a_1 > 0$ depending only on ρ such that

$$|\partial_t \kappa| \leq a_1 (M_{t_m})^{3/2}$$

since

$$|\partial_t k| = \left(\partial_s^2 k + k^3 - k\tau^2\right) < \sqrt{c_3 M_{t_n}^3} + 2\left(c_1 M_{t_n}\right)^{3/2}$$

by growth bound part 2.

As $k^2(p_m, t_m) \geq pM_{t_n}$,

$$k(p_m, t) > k(p_m, t_m) - a_1 M_{t_n}^{3/2} t > \sqrt{pM_{t_m}} - a_1 M_{t_m}^{3/2} t$$

Therefore, there exists $a_2 > 0$ such that

$$k(p_n, t) \geq \frac{|k_{p_m}, t_m|}{\sqrt{2}}$$

for

$$|t - t_n| \leq \frac{a_2}{M_{t_m}}$$

$\exists a_3$ such that

$$(\partial_s \kappa)^2 \leq a_3 (M_{t_m})^2$$

$$|\partial_s \kappa| \leq \sqrt{a_3} M_{t_m}$$

Therefore

$$k(p, t) \geq k(p_m, t) - \sqrt{a_3} M_{t_m} t$$

Finally,

$$\exists d_1 \text{ such that } k(p,t) > \frac{k(p_m,t_m)}{2} \text{ when } (p,t) \in N(p_m,t_m,d)$$

8.2. **Torsion Dissipation.** We now do the exact same thing for first torsion and obtain neighborhoods in which

$$\tau_1(p,t) > \frac{\tau_1(p_m,t_m)}{2}$$

$$\exists \delta > 0 \text{ such that } \forall (p,t) \in N(p_n,t_n,d)$$

$$\tau(p,t) > \frac{\tau(p_n,t_n)}{2}$$

From τ_1 evolution

$$\partial_t \tau_1 = \partial_s^2 \tau_1 + \frac{2}{\kappa}\partial_s \kappa + \partial_s \tau_1 + \frac{2\tau_1}{\kappa}\left(\partial_s^2 \kappa - \frac{1}{\kappa}(\partial_s \kappa)^2 + \kappa^3\right) - \tau_1 \tau_2^2 \tag{23}$$

Starting with Frenet frame calculations,

$$\begin{aligned} c_4 (M_{t_m})^4 &\geq |\partial_s^4 T|^2 \\ \left|\partial_s^4 T\right|^2 &\geq \left(\kappa \partial_s^2 \tau_1 + 3\partial_s \kappa \partial_s \tau_1 + 3\partial_s^2 \kappa \tau_1 - \kappa^3 \tau_1 \kappa \tau_1^3 - \kappa \tau_1 \tau_2^2\right)^2 \\ &= \left(\kappa \partial_t \tau_1 + \partial_s \kappa \partial_s \tau_1 + \frac{\tau_1}{\kappa}\partial_s^2 \kappa + \frac{(\partial_s \kappa)^2 \tau_1}{\kappa} - 2\tau_1 \kappa^2 - \kappa^3 \tau_1 - \kappa \tau_1^3\right)^2 \end{aligned}$$

We want to bound all of the terms in other than $\kappa \partial_t \tau_1$ by terms no larger than $\alpha (M_{t_m})^2$ and then apply Young's inequality to bound $\kappa \partial_t \tau_1$.

We start with the second term

$$(\partial_s k)^2 \leq c_2 (M_{t_m})^2$$

$$k\tau \leq \sqrt{c_2 (M_{t_m})^2}$$

By curvature dissipation,

$$k^2 \geq \frac{\rho M_{t_m}}{2}$$

Applying these equations, we find

$$\frac{2c_2^{3/2}(M_{t_m})^3}{\rho M_{t_m}} \geq \frac{(\partial_s \kappa)^2 \tau_1}{\kappa}. \tag{24}$$

and so

$$\frac{(\partial_s \kappa)^2 \tau_1}{\kappa} \leq \alpha (M_{t_m})^2$$

for some α.

The other terms can be bounded using the same technique of plugging into the Altschuler bounds and then dividing out by a curvature terms if necessary. For conciseness, we leave the rest of this calculation to the reader.

Therefore, there exists some α such that

$$c_4 (M_{t_m})^4 \geq |\partial_s^4 T|^2 \geq \left(\kappa \partial_t \tau_1 + \alpha (M_{t_m})^2\right)^2$$

Therefore,

$$|\kappa \partial_t \tau_1| \leq |1 - \alpha| (M_{t_m})^2$$

Therefore, there exists an $e_3 > 0$ such that $\partial_t \tau_1 > -e_3 (M_{t_m})^{3/2}$ and so exists $g_4 > 0$ such that

$$\tau_1(p_m, t) \geq \frac{1}{\sqrt{2}} \tau_1(p_m, t_m)$$

for $t \in [t_m, t_m + g_4 / M_{t_m}]$

We can similarly bound $\partial_s \tau_1$ since we know that

$$c_3 (M_{t_m})^3 \geq (2\tau_1 \partial_s \kappa + \kappa \partial_s \tau_1)^2$$

and

$$2\tau_1 \partial_s \kappa \leq \frac{4 c_2 M_{t_m}}{\rho \sqrt{M_{t_m}}}$$

Following the same procedure as before, we can finish the proof. For torsion, we also want to bound torsion from above but this follows immediately from the bounds from below since they come from bounding terms in the Taylor's expansion. We could also easily bound torsion from above by $\sqrt{c_2 M_{t_m} / \rho}$ but this approach gives a nicer constant.

We use the smaller of the two dissipation radii (for curvature and first torsion) or $3\rho/64$ to obtain neighborhoods $\mathbf{N}(p_m, t_m, d)$ in which

$$\kappa(p,t) > \frac{\kappa(p_m, t_m)}{2} \text{ and } 2\tau_1(p_m, t_m) > \tau_1(p,t) > \frac{\tau_1(p_m, t_m)}{2}$$

.

9. Planarity

We have obtained all of the bounds that we need for now, we can move on to actually showing that curves become roughly planar as they become singular. At this point, the n-dimensional proof is technically finished because the rest of Altschuler's results hold in n dimensions as in three.

The technique to show that along a singularity a curve becomes flat basically follows from taking the derivative of total curvature. Since total curvature is continuous, if we integrate the derivative of total curvature over a very short amount of time (in one of the neighborhoods we defined earlier), we can make the result arbitrarily small, even as we converge to a singularity. We had a heuristic argument earlier to give a lower bound of $\int_{\mathbf{N}}(p_m, t_m, d)\kappa^3$, so we can show that $(\tau_1/\kappa)^2$ goes to zero along an essential blow-up sequence. We must now do this precisely.

9.1. **Total Curvature Evolution.** Let $K = \int_\gamma \kappa ds$ be the total curvature of γ. Then

$$\partial_t K = \partial_t \left(\int_\gamma \kappa ds \right) = \partial_t \left(\int_0^{2\pi} \kappa \cdot v du \right).$$

Since u and t are independent variables,

$$\partial_t \left(\int_0^{2\pi} \kappa \cdot v du \right) = \int_0^{2\pi} \partial_t(\kappa \cdot v) du.$$

$$\begin{aligned} \int_0^{2\pi} \partial_t(\kappa \cdot v) du &= \int_0^{2\pi} \kappa(\partial_t v) + v(\partial_t \kappa) du \\ &= \int_0^{2\pi} -\kappa^3 v + v\partial_s^2 \kappa + v\kappa^3 - v\kappa(\tau_1^2) du \\ &= \int_\gamma \partial_s^2 \kappa - \kappa(\tau_1^2) ds \end{aligned}$$

For now we assume that there are no inflection points so that this integral becomes

$$-\int_\gamma \kappa(\tau_1^2) ds + \partial_s \kappa|_{\partial\gamma} = -\int_\gamma \kappa(\tau_1^2) ds$$

Total curvature is always decreasing and is bounded above by the total curvature at $t = 0$ and below by 0. Since κ is a continuous function in time so is uniformly continuous. Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_{\theta}^{\theta+\delta} \int_{\gamma} \kappa(\tau_1^2) ds\ dt < \epsilon$$

for any time θ.

$$\int_{N(p_m,t_m,d)} \kappa(\tau_1^2) ds\ dt < \int_{t_m}^{t_m+\frac{d}{M_{t_m}}} \int_{\gamma} \kappa(\tau_1^2) ds\ dt$$

since $N(p_m, t_m, d)$ is a subset of γ during this time. Therefore,

$$\lim_{m\to\infty} \int_{N(p_m,t_m,d)} \kappa(\tau_1^2) ds\ dt = 0.$$

In this calculation we had assumed that there are no inflection points. Torsion is not defined at inflection points so this calculation seems to be flawed if the curve has any. There are a few ways to resolve this issue. A quick argument is that to observe inflection points happen at isolated points in time by the real analyticity of the solution. Therefore, as long as the curve is not planar to begin with (in which case we are finished), on a short time interval any inflection points that appear immediately disappear since the curvature is real analytic. As such, they are a set of zero measure and so will not interfere with our integral. Since we do not assume that torsion is bounded, the fact that torsion blows up around inflection points does not cause a problem either. There are far more illuminating ways of circumventing this but they are irrelevant to what we are trying to do.

We have now completed the first part of the planarity argument. What is left is to establish a lower bound on $\int_{\mathbf{N}(p_m,t_m,d)} \kappa^3 dsdt$. Since κ is roughly M_{t_m} and the size of $\mathbf{N}$ is roughly $\left(\frac{d}{M_{t_m}}\right)^3$, we intuitively expect that this can be done.

What could go wrong with this idea? There are two potential problems. The first is if the curvature or torsion decays extremely quickly from their maximum value then this will not work. However, we have already ruled this out with our dissipation estimates. The second is subtler. If the arc-length were to become small very quickly, then it is possible for the integral of curvature cubed to become very small. We rule this out by showing that distances cannot collapse too quickly. With these two facts (and our ρ value for the sequence), we can finish the proof.

9.2. **Distance Collapse.** Given two points u_1 and u_2 on γ at time t_0, $\text{dist}(u_1, u_2, t_0) = \int_{u_1}^{u_2} ds$.

$$\partial_t \ \text{dist}(u_1, u_2, t) = -\int_{u_1}^{u_2} \kappa^2(s,t)ds$$

$$\geq -\int_{u_1}^{u_2} M_t ds$$

$$\geq -2M_{t_m} \int_{u_1}^{u_2} ds = -2M_{t_m} \cdot \text{dist}(u_1, u_2, t)$$

Integrating this inequality for $t \in [t_m, t_m + \frac{1}{8M_{t_m}}]$,

$$\text{dist}(u_1, u_2, t) \geq \text{dist}(u_1, u_2, t_m) e^{-2M_{t_m}(t-t_m)}$$

On this interval, this implies,

$$\text{dist}(u_1, u_2, t) \geq \text{dist}(u_1, u_2, t_m) e^{-1/4}.$$

9.3. **An Integral Bound and Rough Planarity.** We are now ready to prove the rough planarity theorem from the section on singularities, which we restate here.

Theorem. *Given an essential blow-up sequence (p_m, t_m),*

$$\lim_{m\to\infty} \left(\frac{\tau_1}{\kappa}(p_m, t_m)\right)^2 = 0$$

Proof. By the previous work on dissipation estimates,

$$\liminf_{m\to\infty} \int_{\mathbf{N}(p_m,t_m,d)} \kappa^3 \ ds \ dt \geq \left(\frac{\rho M_{t_m}}{2}\right)^{3/2} \int_{\mathbf{N}(p_m,t_m,d)} ds \ dt.$$

Then, by the distance collapse results, we know that

$$\int_{\mathbf{N}(p_m,t_m,d)} ds \ dt \geq \left(\frac{d}{M_{t_m}}\right)^{3/2} \cdot e^{-1/4}.$$

Combining these two results

$$\begin{aligned}\liminf_{m\to\infty}\int_{N(p_m,t_m,d)}\kappa^3\ ds\ dt \quad &\geq \quad \left(\frac{\rho M_{t_m}}{2}\right)^{3/2}\cdot\left(\frac{d}{M_{t_m}}\right)^{3/2}\cdot e^{-1/4}\\ &= \quad \left(\frac{\rho d}{2}\right)^{3/2}\cdot e^{-1/4}.\end{aligned}$$

However, by our dissipation estimates, we know that in this neighborhood (and so along this essential sequence)

$$\limsup_{m\to\infty}\left(\frac{\tau_1}{\kappa}\right)^2 \leq \limsup_{m\to\infty}\frac{16\cdot\int_{\mathbf{N}(p_m,t_m,d)}\kappa\cdot(\tau_1^2)\,ds\,dt}{\int_{\mathbf{N}(p_m,t_m,d)}\kappa^3\,ds\,dt} \leq \frac{0}{\left(\frac{\rho d}{2}\right)^{3/2}\cdot e^{-1/4}}$$

□

9.4. **Planarity.** We give Altschuler's proof that shows that the curves become planar in a stronger sense as well. Altschuler notes that given $\gamma_t(s)$ a solution of curve shortening flow, if one dilates space by a factor of λ and time by a factor of λ^2, one obtains another solution to the flow. Furthermore, any motion of $\gamma_t(s)$ gives a solution to curve shortening flow as well. Therefore, given an essential blow-up sequence (p_m, t_m), if one rescales space by λ_m so that $\kappa(p_m, t_m) = 1$ for each m, and translates and rotates $\lambda_m \cdot \gamma_t(s)$ to obtain the curve γ^m with $\gamma_t^m(p_m) = 0$, $T_m(p_m, 0) = e_1$ and $N_m(p_m, 0) = e_2$.

For $\tilde{t} = \lambda_m^2(t - t_m)$,

$$\partial_{\tilde{t}}\gamma_{\tilde{t}}^m = \kappa N$$

(i.e. $\gamma_{\tilde{t}}^m$ is a solution to curve shortening flow) on the interval

$$[-\lambda_m^2 t_m,\ \lambda_m^2\cdot(\omega - t_m)).$$

These γ^m are therefore called renormalized solutions to curve shortening flow. Since M_{t_m} goes to ∞, λ_m does and therefore $-\lambda_m^2 t_m$ goes to $-\infty$. We now show that there exists a subsequence $\gamma_t^{m_k}$ which converges to a smooth limiting curve γ^∞. This curve γ^∞ is a solution to curve shortening flow on the time interval $[-\infty, 0]$ and some forward time as well and so is called an ancient solution of the flow.

There is no guarantee that γ^∞ is a compact loop so we parametrize the curves λ^m by their arclength from the origin.

We define the operator

$$\delta_t = \partial_t + \phi_m(s)\partial_s$$

where

$$\phi_m(s) = \int_0^s \kappa^2 ds$$

Then:

$$\begin{aligned}
[\delta_t, \partial_s] &= \partial_t\partial_s + \phi_m\partial_s\partial_s - \partial_s\partial_t - (\partial_s\phi_m)\partial_s - \phi_m\partial_s\partial_s \\
&= \kappa_m^2\partial_s - \kappa_m^2\partial_s \\
&= 0
\end{aligned}$$

Furthermore, since (p_m, t_m) is an essential sequence, $\rho \cdot \sup k_m^2 \leq 1$ and so by the earlier bounding results, we know

$$|\partial_s^n T_m|^2 < c_n \text{ for } t < 0$$

for all n.

Therefore, $|\partial_t^n T_m|^2$ is also bounded for all n as each one can be expressed in terms of $|\partial_s^n T_m|^2$. Finally, since, $\phi(s) \leq s/\rho$, we know that $|\delta_{t^n} T_m|^2$ is bounded for all n and so for any compact subset of $\mathbb{R}\times[-\lambda_m^2 t_m,\ \lambda_m^2\cdot(\omega - t_m))$, we have bounds on all of the mixed operators $|\delta_{t^n}\partial_s^{n'} T|^2$ (independent of m). Since the operators commute, we only need to consider this combination.

Finally, we appeal to the Ascoli-Arzela theorem to show that there is a subsequence (p_{m_k}, t_{m_k}) on which the tangent vectors converge uniformly for any compact set in $\mathbb{R} \times [-\infty,\ \lim_{m\to\infty} \lambda_m^2 \cdot (\omega - t_m))$. By integrating these tangent vectors, we obtain the limiting curve γ^∞. If the limiting curve is a closed loop, we just consider one period of this curve. Notice that this curve is not a line since the curvature at the origin is 1.

We now show that γ is a planar curve. By our integral estimate, we know that for all $N > 0$ and $\epsilon > 0$,

$$\int_{-N}^{0}\int_{\gamma^\infty} \kappa \cdot \tau_1^2 \, ds \, dt < \epsilon$$

and so $\tau_1 = 0$ whenever $\kappa \neq 0$ in γ^∞. Since γ^∞ is real-analytic, this shows that γ^∞ actually lies in a single plane. This implies that the higher torsions also vanish for the renormalized curve.

10. Classification of Singularities

Altschuler's work goes further than this. He uses theorems by Angenent, Huisken and Hamilton to specify what the possible shapes of the limiting curves are. The geometry of these solutions is very interesting and we discuss this further, although we will not provide the proofs of the next two theorems here but refer the reader to Altschuler's work.

To continue we need a few definitions. A blow-up singularity is Type I if $\lim_{t\to\omega} M_t \cdot (\omega - t)$ is bounded and Type II otherwise. In the planar case, the simplest example of a Type I singularity is the curve approaching a circle as it collapses to a point. The simplest example of a type II singularity is of a cusp emerging. Examples of curves that do this are Cayley's Sextet, a figure eight curve or a Limaçon with an inner loop.

We simply state the following theorems:

Theorem 7. *Given a curve which develops a type I singularity, every blow-up sequence is essential and the curve approaches a homothetically shrinking (self-similar) curve studied by Abresch and Langer. For these singularities, all blow-up sequences are essential.*

Theorem 8. *Given a curve which develops a type II singularity, there exists an essential sequence whose renormalized limit is the planar curve* $y = -log(cos(x))$*, which is known as the Grim Reaper Curve.*

These are two very powerful theorems and we refer the reader to [2] for their proof. This essentially classifies any possible singularity of curve shortening flow. The Grim Reaper Curve is an interesting curve when studied independently. Its solution to curve shortening flow exists for all positive and negative time and flows by translating it across the plane.

One question remains. When do the first type of singularities occur and when do the second occur? There are several observations that can be made although there are few theorems available. Grayson showed that any embedded curve in the plane shrinks to a point, but if we are dealing with spacial curves, the situation may become more complicated. We know that there are limiting planar curves that have self-intersections (Abresch-Langer solutions) and not every embedded spacial curve approaches a circle.

However, the two different types of singularities give very different behavior and so being able to give conditions under which one or the other occurs would be a good start.

We first note that given an arbitrary curve it is very difficult to tell which type of singularity the curve is approaching because we have no way to a priori way of knowing how far the curve is from developing a singularity. Therefore, we must try to find quantities that distinguish these types. Fortunately, Altschuler's results are very powerful and give us several quantities that display these behaviors.

Directly from the definitions, we know that given $l > 0$ and a blow-up sequence (p_m, t_m)

$$\lim_{m \to \infty} \frac{M_{\left(t_m + \frac{l}{M_{t_m}}\right)}}{M_{t_m}} = 1$$

for Type II singularities and is either greater than 1 or undefined (if l is too large) for Type I singularities.

Another potentially useful quantity comes from studying the modified distortion of the curve. The distortion of a curve was introduced by Mikhael Gromov in [9] and is often studied in the context of knot theory. Since embedded curves in $\mathbb{R}^n$ can intersect when shortening by (1), the standard distortion of a curve, which is defined as $\sup_{p,p' \in \gamma} \mathrm{dist}_{int.}(p,p')/\mathrm{dist}_{Euc.}(p,p')$ can blow up before the curve becomes singular. In view of this, we modify the definition slightly to obtain a useful quantity in this context. Given $p \in [0, 2\pi]$, let $f(p) = p'$ where $\int_p^{p'} \kappa(\gamma(u)) \cdot ||\dot{\gamma}|| du = \pi$.

Let

$$D(\gamma) = \sup_{p \in [0,2\pi]} \frac{\mathrm{dist}_{int.}(p, f(p))}{\mathrm{dist}_{Euc.}(p, f(p))}$$

This quantity is scale invariant by examining the Grim Reaper curve, we know that given a type II singularity

$$\lim_{t\to\omega} D(\gamma(t)) = \infty.$$

Furthermore, utilizing a lemma by A. Schur and E. Schmidt (Lemma 3.2.3 in [7]), we know that

$$D(\gamma(t)) < M_t \cdot L(\gamma(t))$$

where $L(\gamma(t))$ is the length of the curve. Whenever a type I singularity emerges, the latter quantity is bounded (since the limiting solution is homothetically shrinking that quantity is scale invariant). Therefore, both $D(\gamma(t))$ and $M_t \cdot L(\gamma(t))$ are bounded for type I singularities.

If one were able to find initial conditions which give control over any of these quantities, then under those initial conditions, one would be able to determine what sort of singularity develops. Using this idea, we have the following conjecture.

Conjecture 9. *Given a curve γ that is initially embedded, it develops a type I singularity.*

Consider an embedded curve in R^n that has a segment which approaches a Grim Reaper Curve and so develops a type II singularity. At some point of the blow up sequence, the segment will be very similar to the Grim Reaper Curve and evolve in a similar way. We renormalize the curve along this segment at some finite step of an essential blow up sequence on this segment. Now if one considers the normal vectors just past the endpoints of this segment (for which the total curvature between the two points is exactly pi), if the segment lies entirely in one plane, then by real analyticity of the curve, we know the curve is globally planar and so can apply the results from [8]. However, if the normal vectors point away from each other just past the endpoints of the segment, the modified distortion decreases on the segment, and if the normal vectors point out of the plane, torsion is introduced on the section approaching the Grim Reaper Curve, which contradicts the planarity theorem (since we are considering the renormalized curve). This heuristic is incomplete for several reasons. It lacks a formal argument ruling out the normal vectors pointing towards the inside of the curve but slightly out of the plane. It might be possible to generalize the delta-whisker lemma from [8] for when the curve is very close to being planar and the normal vectors are close to lying in the plane. Since spatial curves can intersect each other (but only do so on a discrete set), this seems to be the greatest challenge. Furthermore, the argument showing that normal vectors pointing out of the plane would introduce torsion on the segment must be made more precise although one could probably do this using a short time translation and obtaining the a blow up sequence which contradicts the planarity theorem. It is our hope that this argument can be formalized because it would show that a dense open set of curves in R^n for $n > 2$ have type I singularities and that type II singularities, whose archetype is an emerging cusp, are often avoided.

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